PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Spherical Gravitational Waves

W. B. Bonnor

Phil. Trans. R. Soc. Lond. A 1959 **251**, 233-271
doi: 10.1098/rsta.1959.0003

SPHERICAL GRAVITATIONAL WAVES

By W. B. BONNOR, Ph.D.
Queen Elizabeth College (University of London), Campden Hill Road, London, W. 8

(*Communicated by L. Rosenhead, F.R.S.—Received* 12 *March* 1958—*Revised* 4 *July* 1958)

CONTENTS

The field of gravitational radiation emitted from two moving particles is investigated by means of general relativity. A method of approximation is used, and in the linear approximation retarded potentials corresponding to spherical gravitational waves are introduced. As is already known, the theory in this approximation predicts that energy is lost by the system. The field equations in the second, non-linear, approximation are then considered, and it is shown that the system loses an amount of gravitational mass precisely equal to the energy carried away by the spherical waves of the linear approximation. The result is established for a large class of particle motions, but it has not been possible to determine whether energy is lost in free gravitational motion under no external forces.

The main conclusion of this work is that, contrary to opinions frequently expressed, gravitational radiation has a real physical existence, and in particular, carries energy away from the sources.

1. Introduction

It has long been known that the field equations of general relativity,

$$R_{ik} = 0, \tag{1·1}$$

admit, in the linear approximation, solutions which refer to gravitational radiation. This radiation seems in many respects similar to that of electromagnetism; in particular, it is transmitted with the speed of light, carries energy away from the source, and at great distance r exerts a force proportional to r^{-1}. According to the approximate theory, waves should be emitted during the motion of bodies in a gravitational field whether this motion takes place freely, or under the action of non-gravitational forces. The rate of loss of energy for the motions of astronomical bodies can be estimated, and for stars and planets this turns out to be below the present limit of detection (Landau & Lifshitz 1951).

In the early history of general relativity these results were accepted at their face value, though no doubt it was recognized that owing to the non-linearity of (1·1), there must be certain reservations about conclusions drawn from the linear approximation. In 1938 a method of approximation was given by Einstein, Infeld & Hoffmann (E. I. & H.) which dealt successfully with the non-linearity and enabled the equations of free gravitational motion of particles to be obtained up to, in principle, any approximation. In the method of E. I. & H., and in the later refinements of it, there is no sign of loss of energy due to gravitational radiation.

This result tended to cast doubt on the physical reality of gravitational waves—though strictly it applied only to free gravitational motion, not to all motion. In the light of the work of E. I. & H. it was easy to point to flaws in the linear theory which predicts radiation: apart from the neglect of the non-linear terms in (1·1), one may object to the energy pseudo tensor, used to calculate the rate of loss of energy, on the grounds that it has no covariant meaning.

The controversy would be settled by the discovery of a suitable exact solution of (1·1). A number of exact wave-like solutions are known, and these add a certain plausibility to the physical existence of gravitational waves (Einstein & Rosen 1937; Takeno 1956; Bonnor 1957; Bondi 1957; Weber & Wheeler 1957). However, they do not settle the question because none of them makes it clear what the sources of the waves are. A proper understanding of the behaviour of the sources is essential because if the waves carry away energy, the sources should lose a corresponding amount of mass.

In this paper I attack the problem by a method of approximation which is different from that of E. I. & H. Basically my method consists merely in proceeding from the ordinary linear approximation to the simpler non-linear ones. In the linear approximation I use a solution which is a model of two equal particles moving symmetrically in a straight line under the action of a spring or some other machine. At this stage I insert, as one is entitled to do, the field of outgoing spherical waves. As is already known, no loss of mass appears in the first approximation, but if the waves really carry energy a loss of mass should appear in the second approximation. It turns out that *there is in the second approximation a loss of gravitational mass which is precisely equal to the energy carried away by the waves inserted in the linear approximation.*

As will be explained in § 12, it has not been possible to determine whether mass is lost if the particles are moving freely under their own gravitation. The result of E. I. & H.—that mass is not lost—may well depend on their use of the average of advanced and retarded potentials. In accordance with the usual physical arguments, the use of retarded potentials is more realistic for isolated systems, so it seems that a satisfactory solution to the problem of free motion must await a method of approximation which uses retarded potentials.

Other writings which bear on this work are those of Rosen & Shamir (1957), and of Fock (1957). Rosen & Shamir consider the first approximation to (1·1) in the same co-ordinate system as mine, but do not proceed to the non-linear approximations. The work of Fock, which is as yet known to me only through a short review article (1957), seems to reach conclusions in some respects similar to mine, though by a different method.

Many of the calculations in this work are long and tedious, and as far as possible these are relegated to Appendices. The plan of the paper is as follows. The physical system of interest is described in § 2, and the solution of the wave equation appropriate to it is given in § 3 (and appendix I); § 4 deals with the method of approximation, and § 5 with the metric. After a brief section on notation (§ 6), the solutions of the necessary linear approximations are given in § 7 (and appendices II and III). A discussion preliminary to the non-linear approximation occurs in § 8, and this includes further consideration of the physical system under investigation. The longest and most difficult section, § 9, gives the solution to the non-linear approximation; readers not interested in the details may prefer to omit this section except for the summary in the last paragraph. Sections 10, 11 and 12 examine the implications of the solution and derive the main results. The paper ends with some remarks on the method of approximation (§ 13) and with a Conclusion.

2. The physical system

I shall construct an approximate solution for two particles A, B of equal mass m moving symmetrically in the straight line AB about their middle point O (figure 1). For the time being we may think of distance and time as having their Newtonian meaning.

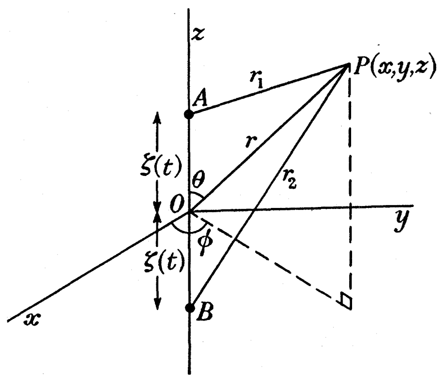

Figure 1

The notation used is as follows. The origin of co-ordinates is taken as O, and the axis of z is along BA. The co-ordinates of the particles are therefore

$$A = (0, 0, \zeta(t)), \quad B = (0, 0, -\zeta(t)),$$

where $\zeta(t)$ is a given function of the time t. Let $P(x, y, z)$ be a field-point, fixed relative to the co-ordinate axes. Write

$$AP = r_1, \quad BP = r_2, \quad OP = r,$$

as in figure 1; then

$$\left.\begin{aligned} r_1^2 &= x^2+y^2+(z-\zeta)^2 = r^2-2\zeta z+\zeta^2, \\ r_2^2 &= x^2+y^2+(z+\zeta)^2 = r^2+2\zeta z+\zeta^2. \end{aligned}\right\} \tag{2·1}$$

We may also give P spherical polar co-ordinates (r, θ, ϕ), θ being the angle AOP and ϕ the usual azimuthal angle.

The reason why we shall consider two equal particles moving symmetrically is as follows. If the particles were unequal or the motion unsymmetrical, it would be possible to distinguish a positive direction along Oz and a right-handed screw could be defined. One might expect in these circumstances that a component of the wave field would point in the ϕ-direction: for example, there might be a component of the gravitational field perpendicular to the plane zOx. Such a component of the electromagnetic field appears in the case of an oscillating electric dipole with charges $\pm e$. This component would destroy the axial symmetry of the field, and would complicate the metric which we shall use in § 5.

Having made this choice of symmetry, which reduces to zero the dipole moment of the particles about the origin, we expect not to find a 'dipole wave', i.e. one in which the potentials at P are proportional to $\cos\theta$. This expectation is confirmed in § 7.

We do not expect the particles A and B to describe the arbitrary motion given by $\zeta(t)$ without the application of some external forces. These forces we suppose to be supplied by some machine, which may be something like the ideal elastic spring of Newtonian theory, or may be some electrical or other device giving energy to the particles. We do not need to specify it further, except to require it to have axial symmetry, and symmetry about the plane $z = 0$; we also suppose that it is confined within a finite region near the origin. We shall refer to the machine again in § 8.

3. The solution of the wave equation

As shown in the standard textbooks (Eddington 1924) the linear approximation to equations (1·1) involves, in a certain co-ordinate system, the ordinary wave equation. This equation also appears in the co-ordinate system which I shall use, so I give here the solution of it which will be needed for this problem.

Using the ordinary three-dimensional notation of § 2, a solution of the wave equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=0 \tag{3·1}$$

is

$$V=\left[\frac{m}{r_1-(\mathbf{v}_1\cdot\mathbf{r}_1)/c}\right]. \tag{3·2}$$

Here m denotes any quantity associated with the point-source A which is moving with velocity $\mathbf{v}_1$, $\mathbf{r}_1$ has the meaning given to it in § 2, and the square brackets mean that the quantities inside them are to be calculated at time $t-r_1/c$. In electromagnetism, if m is replaced by the charge e of the source, the expression (3·2) represents the scalar potential of the charge at the field-point $P(x, y, z)$ at time t. I shall take (3.2) as the solution of the wave equation (3·1) corresponding to our problem, and shall suppose that m refers to the mass of the particle.

It is not known at this stage whether the emission of gravitational waves results in a change in mass of the moving particles, and so one has no idea what function of $t-r_1/c$ is to be chosen to represent m in (3·2). The simplest procedure will be to take m as constant and to allow any change in mass to appear in the form of correction terms in the course of the approximation method. Accordingly the solution of (3·1) corresponding to the particle at A will be chosen as

$$V_1=\frac{m}{[r_1-(\mathbf{v}_1\cdot\mathbf{r}_1)/c]}, \tag{3·3}$$

where m is the mass, provisionally taken as constant.

The solution (3·3) is not convenient when more than one particle is present. It will be more suitable to express it as a function of $t-r/c$, where $r = OP$, as in § 2. To do this, let us introduce g by

$$g=r_1-r=(r^2-2z\zeta+\zeta^2)^{\frac{1}{2}}-r; \tag{3·4}$$

then it is shown in appendix I that

$$V_1 = \frac{m}{r_1} + \sum_{n=1}^{\infty} \left(-\frac{1}{c}\right)^n \frac{m}{n!} \frac{\partial^n}{\partial t^n} \left(\frac{g^n}{r_1}\right), \tag{3·5}$$

where r_1 and g are to be taken at time $t - r/c$.

Let us write

$$\zeta(t) = af(t), \tag{3·6}$$

where a is a constant with the dimensions of a length and $f(t)$ is independent of a and m; then since $\zeta(t)$ is a length, $f(t)$ is dimensionless. If we now suppose that V_1 in (3·5) can be expanded in a power series in a we have (appendix I)

$$V_1 = \frac{m}{r} + m \sum_{n=1}^{\infty} a^n F_n(r, \theta, t - r/c), \tag{3·7}$$

where $F_n(r, \theta, t - r/c)$ are functions determined in the course of the expansion.

We now consider the system composed of the two equal particles A and B moving symmetrically, and we find that odd powers of a do not appear in the series for the total potential which we write as

$$V = \frac{2m}{r} + 2m \sum_{s=1}^{\infty} a^{2s} G_{2s}(r, \theta, t - r/c). \tag{3·8}$$

The G_{2s} may be calculated from (3·5) and the corresponding equation for the second particle. In the expansion it is necessary to assume that r is greater than the maximum value of $|\zeta|$. The values of G_2 and G_4 are (appendix I):

$$G_2 = P_2 \left[\frac{f^2}{r^3} + \frac{(f^2)'}{cr^2} + \frac{(f^2)''}{3c^2 r}\right] + \frac{(f^2)''}{6c^2 r}, \tag{3·9}$$

$$G_4 = P_4 \left[\frac{f^4}{r^5} + \frac{(f^4)'}{cr^4} + \frac{3(f^4)''}{7c^2 r^3} + \frac{2(f^4)'''}{21c^3 r^2} + \frac{(f^4)^{\text{iv}}}{105c^4 r}\right]$$
$$+ P_2 \left[\frac{(f^4)''}{14c^2 r^3} + \frac{(f^4)'''}{14c^3 r^2} + \frac{(f^4)^{\text{iv}}}{42c^4 r}\right] + \frac{(f^4)^{\text{iv}}}{120c^4 r}, \tag{3·10}$$

where $'$ means $\mathrm{d}/\mathrm{d}t$, P_n are the Legendre polynomials, and f^2 and all its derivatives are to be calculated at time $t - r/c$. The function f is the dimensionless function introduced in (3·6), and r, it may be repeated, is the radius vector OP, from the origin to the field-point.

The solution of form (3·8), with the first two coefficients given by (3·9) and (3·10) will be needed in § 7. There, and in the rest of the paper, except where stated, c is taken as 1.

4. The method of approximation

The system of symmetrically moving masses described in §§ 2 and 3 involves the parameters m and a^2. The solution of (1·1) corresponding to the system will contain these parameters, and we may suppose that there is a double infinity of solutions obtained by varying them. Let us assume that the components of the metric tensor g_{ik} for this family of solutions can be expanded in a convergent power series in terms of the parameters m and a^2:

$$g_{ik} = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} m^p (a^2)^s \overset{(ps)}{g}_{ik}, \tag{4·1}$$

where $\overset{(ps)}{g}_{ik}$ involves x, y, z and t, but not m or a^2. Then if (4·1) is substituted into the field equations (1·1) we shall obtain zero for the coefficients of powers of m, a^2 and of products of these powers.

I shall use this method of expansion in terms of two parameters to obtain an approximate solution of the field equations (1·1). The first few terms of (4·1) written out in full are

$$\left.\begin{aligned} g_{ik} = \overset{(00)}{g}_{ik} + a^2\overset{(01)}{g}_{ik} + a^4\overset{(02)}{g}_{ik} + \dots \\ + m\overset{(10)}{g}_{ik} + m^2\overset{(20)}{g}_{ik} + \dots \\ + ma^2\overset{(11)}{g}_{ik} + ma^4\overset{(12)}{g}_{ik} + \dots \\ + m^2a^2\overset{(21)}{g}_{ik} + m^2a^4\overset{(22)}{g}_{ik} + \dots . \end{aligned}\right\} \tag{4·2}$$

We are supposing that the field depends essentially on the existence of the two masses m so that if these are absent the space-time will be flat. Hence, all the terms on the right-hand side of (4·2) (except the first, which refers to flat space-time) must involve m, and so

$$\overset{(0s)}{g}_{ik} = 0 \quad (s > 0). \tag{4·3}$$

If, on the other hand, $a = 0$ we have the field of a particle of mass $2m$: hence we may take

$$\overset{(p0)}{g}_{ik} = \text{Schwarzschild field of a mass } 2m \quad (p \geqslant 0).$$

In supposing this we do not rule out the possibility of a solution in which mass is lost as radiation energy; terms corresponding to loss of mass may still appear from further stages in the approximation.

The first stage of the approximation to involve a wave field comes from the third row on the right of (4·2). This is clear from (3·8), in which the wave-like terms occur in the G_{2s}, that is, in the coefficients of ma^{2s}. Thus for $\overset{(1s)}{g}_{ik}$ we shall take expressions like G_{2s}, such as (3·9) and (3·10). (In the co-ordinate system which we shall choose not all the $\overset{(1s)}{g}_{ik}$ satisfy the wave equation; however, it turns out that $\overset{(1s)}{g}_{11}$ does, and the others can be calculated in terms of it.)

The higher terms in (4·2), such as those in the fourth row, are interaction terms resulting from the lower ones. Of these we shall consider in detail only one—that involving m^2a^4. The field equations to be satisfied at these stages involve non-linear terms coming from the previous approximations, and it is here that the loss of energy by the sources may be expected to show in the coefficients of the metric.

5. The metric

It is convenient to use spherical polar co-ordinates so that the metric of flat space-time is

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + dt^2,$$

and the values of the non-zero $\overset{(00)}{g}_{ik}$ are

$$\overset{(00)}{g}_{11} = -1, \quad \overset{(00)}{g}_{22} = -r^2, \quad \overset{(00)}{g}_{33} = -r^2\sin^2\theta, \quad \overset{(00)}{g}_{44} = 1.$$

For the reasons given in § 2 we assume that the problem of the symmetrically oscillating particles is one with complete symmetry about Oz, and that the metric may be written in the form

$$ds^2 = -A\,dr^2 - Br^2\,d\theta^2 - Cr^2\sin^2\theta\,d\phi^2 + D\,dt^2 + 2E\,dr\,d\theta + 2F\,dr\,dt + 2G\,d\theta\,dt, \tag{5·1}$$

where $A, B, \ldots, G$ are functions of r, θ and t.

Let us suppose that $A, B, \ldots, G$ can be expanded in a power series in terms of a single parameter λ, as follows:

$$A = 1 + \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{A}, \quad E = \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{E},$$

$$B = 1 + \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{B}, \quad F = \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{F},$$

$$C = 1 + \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{C}, \quad G = \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{G}.$$

$$D = 1 + \sum_{s=1}^{\infty} \lambda^s \overset{(s)}{D},$$

Now carry out the transformation of co-ordinates:

$$\left.\begin{aligned} r &= r^* + \lambda\overset{(1)}{\alpha}(r^*, \theta^*, t^*), \\ \theta &= \theta^* + \lambda\overset{(1)}{\beta}(r^*, \theta^*, t^*), \\ \phi &= \phi^*, \\ t &= t^* + \lambda\overset{(1)}{\delta}(r^*, \theta^*, t^*), \end{aligned}\right\} \tag{5·2}$$

and try to choose the new co-ordinates so that

$$\overset{(1)}{g}{}^*_{12} = \overset{(1)}{g}{}^*_{14} = \overset{(1)}{g}{}^*_{24} = 0.$$

One finds then

$$\overset{(1)}{g}{}^*_{12} = -\overset{(1)}{\alpha}_2 + \overset{(1)}{E} - r^{*2}\overset{(1)}{\beta}_1 = 0, \tag{5·3}$$

$$\overset{(1)}{g}{}^*_{14} = -\overset{(1)}{\alpha}_4 + \overset{(1)}{F} + \overset{(1)}{\delta}_1 = 0, \tag{5·4}$$

$$\overset{(1)}{g}{}^*_{24} = -r^{*2}\overset{(1)}{\beta}_4 + \overset{(1)}{G} + \overset{(1)}{\delta}_2 = 0, \tag{5·5}$$

where the suffixes 1, 2 and 4 on the right mean differentiation with respect to r^*, θ^* and t^*, respectively.

The compatibility condition for equations (5·3) and (5·4) is

$$r^{*2}\overset{(1)}{\beta}_{14} + \overset{(1)}{\delta}_{12} = \overset{(1)}{E}_4 - \overset{(1)}{F}_2. \tag{5·6}$$

This has to be compatible with (5·5); differentiating (5·5) with respect to r^* gives

$$2r^*\overset{(1)}{\beta}_4 + r^{*2}\overset{(1)}{\beta}_{41} - \overset{(1)}{\delta}_{21} = \overset{(1)}{G}_1. \tag{5·7}$$

Equations (5·6) and (5·7) are compatible if

$$2r^{*2}\overset{(1)}{\beta}_{14}+2r^{*}\overset{(1)}{\beta}_{4}=\overset{(1)}{G}_{1}+\overset{(1)}{E}_{4}-\overset{(1)}{F}_{2},$$

which always has a solution for $\overset{(1)}{\beta}$. Hence, we can find $\overset{(1)}{\alpha}$, $\overset{(1)}{\beta}$ and $\overset{(1)}{\delta}$ satisfying (5·3) to (5·5) and thus reduce the first approximation to the metric (5·1) to diagonal form.

Having eliminated $\overset{(1)}{g}_{12}$, $\overset{(1)}{g}_{14}$ and $\overset{(1)}{g}_{24}$, we can proceed in exactly the same way to eliminate $\overset{(2)}{g}_{12}$, $\overset{(2)}{g}_{14}$ and $\overset{(2)}{g}_{24}$ by a transformation of the form

$$r=r^{*}+\lambda^{2}\overset{(2)}{\alpha}(r^{*},\theta^{*},t^{*}),$$

....

Thus provided the metric (5·1) can be expanded in powers of a parameter it can always be reduced to diagonal form.

A similar proof applies, of course, if the functions $A, B, \ldots, G$ are expansible in terms of two parameters. Hence, we may assume for the purposes of our problem that *the metric may be reduced to diagonal form*, which we take to be

$$ds^{2}=-A\,dr^{2}-Br^{2}\,d\theta^{2}-Cr^{2}\sin^{2}\theta\,d\phi^{2}+D\,dt^{2}. \tag{5·8}$$

6. Notation

In developing the successive approximations we shall use the following notation.

The (ps) approximation will mean the coefficient of $m^{p}(a^{2})^{s}$ in (1·1), and the symbols $\overset{(ps)}{g}_{ik}$ will denote the coefficient of $m^{p}(a^{2})^{s}$ in the expansion of g_{ik} as in (4·2). To save writing we shall also use the following, which is an extension of the notation of § 5:

$$\left.\begin{aligned}
-g_{11}&=A=1+\sum_{p=1}^{\infty}\sum_{s=0}^{\infty}m^{p}a^{2s}\overset{(ps)}{A},\\
-g_{22}&=Br^{2}=r^{2}\Big[1+\sum_{p=1}^{\infty}\sum_{s=0}^{\infty}m^{p}a^{2s}\overset{(ps)}{B}\Big],\\
-g_{33}&=Cr^{2}\sin^{2}\theta=r^{2}\sin^{2}\theta\Big[1+\sum_{p=1}^{\infty}\sum_{s=0}^{\infty}m^{p}a^{2s}\overset{(ps)}{C}\Big],\\
g_{44}&=D=1+\sum_{p=1}^{\infty}\sum_{s=0}^{\infty}m^{p}a^{2s}\overset{(ps)}{D}.
\end{aligned}\right\} \tag{6·1}$$

As further abbreviations we shall sometimes use, after giving warning,

$$\overset{(1s)}{A}=\alpha,\quad \overset{(1s)}{B}=\beta,\quad \overset{(1s)}{C}=\gamma,\quad \overset{(1s)}{D}=\delta, \tag{6·2}$$

$$\overset{(22)}{A}=\mu,\quad \overset{(22)}{B}=\nu,\quad \overset{(22)}{C}=\sigma,\quad \overset{(22)}{D}=\rho. \tag{6·3}$$

The quantities α, β and δ in (6·2) have no connexion with the functions in (5·2) which will not be needed further.

A suffix 1, 2 or 4 after a non-tensorial symbol such as $\overset{(ps)}{A}$, α or μ will mean differentiation with respect to r, θ or t.

7. The linear approximations

The solutions of the (00) and (10) approximations are

$$\overset{(00)}{g}_{11} = -1, \quad \overset{(00)}{g}_{22} = -r^2, \quad \overset{(00)}{g}_{33} = -r^2\sin^2\theta, \quad \overset{(00)}{g}_{44} = 1; \tag{7·1}$$

$$\overset{(10)}{g}_{11} = -4r^{-1}, \quad \overset{(10)}{g}_{22} = 0, \quad \overset{(10)}{g}_{33} = 0, \quad \overset{(10)}{g}_{44} = -4r^{-1}. \tag{7·2}$$

These are simply part of the Schwarzschild solution for a particle of mass $2m$, to which the system reduces if $a = 0$.

Considering now the $(1s)$ approximation $(s>0)$, we notice that, since all the terms $\overset{(0s)}{g}_{ik}$ $(s>0)$ are zero (see (4·3)), it is not possible to make up expressions containing ma^{2s} from lower approximations. Hence, the $(1s)$ approximation consists of linear differential equations, homogeneous in the sense that every term in it contains one of the $\overset{(1s)}{g}_{ik}$ or one of their derivatives.

In the notation of (6·2), the equations of the $(1s)$ approximation are (appendix II):

$$R_{11} \equiv \tfrac{1}{2}ma^{2s}[\beta_{11}+\gamma_{11}+\delta_{11}+2r^{-1}(\beta_1+\gamma_1-\alpha_1)+r^{-2}(\alpha_{22}+\alpha_2\cot\theta)-\alpha_{44}] = 0, \tag{7·3}$$

$$R_{22} \equiv \tfrac{1}{2}ma^{2s}r^2[\beta_{11}+r^{-1}(3\beta_1-\alpha_1+\gamma_1+\delta_1)+r^{-2}(2\beta-2\alpha)-\beta_{44} +r^{-2}(\alpha_{22}+\gamma_{22}+\delta_{22}-\beta_2\cot\theta+2\gamma_2\cot\theta)] = 0, \tag{7·4}$$

$$R_{33} \equiv \tfrac{1}{2}ma^{2s}r^2\sin^2\theta[\gamma_{11}+r^{-1}(3\gamma_1-\alpha_1+\beta_1+\delta_1)+r^{-2}(2\beta-2\alpha)-\gamma_{44} +r^{-2}\gamma_{22}+r^{-2}\cot\theta(2\gamma_2-\beta_2+\alpha_2+\delta_2)] = 0, \tag{7·5}$$

$$R_{44} \equiv \tfrac{1}{2}ma^{2s}[\alpha_{44}+\beta_{44}+\gamma_{44}-\delta_{11}-2r^{-1}\delta_1-r^{-2}(\delta_{22}+\delta_2\cot\theta)] = 0, \tag{7·6}$$

$$R_{12} \equiv \tfrac{1}{2}ma^{2s}[\gamma_{12}+\delta_{12}+(\gamma_1-\beta_1)\cot\theta-r^{-1}(\alpha_2+\delta_2)] = 0, \tag{7·7}$$

$$R_{14} \equiv \tfrac{1}{2}ma^{2s}[\beta_{14}+\gamma_{14}+r^{-1}(\beta_4+\gamma_4-2\alpha_4)] = 0, \tag{7·8}$$

$$R_{24} \equiv \tfrac{1}{2}ma^{2s}[\alpha_{24}+\gamma_{24}+\cot\theta\,(\gamma_4-\beta_4)] = 0. \tag{7·9}$$

Equations (7·3) to (7·9) are equivalent to the set calculated by Rosen & Shamir (1957), who considered the linear approximation to (1·1) in the same co-ordinate system as that used here.

None of the equations (7·3) to (7·9) is a wave equation as it stands. However, it is shown in appendix III how one can derive the equation

$$\alpha_{11}+2r^{-1}\alpha_1+r^{-2}(\alpha_{22}+\alpha_2\cot\theta)-\alpha_{44} = \int (v_{11}+r^{-1}v_1)\,\mathrm{d}\theta - w_1 - r^{-1}w - u_1 - r^{-1}u, \tag{7·10}$$

where $u(r,\theta)$, $v(r,\theta)$ and $w(r,t)$ are functions of integration. The functions β, γ and δ are determined in terms of α and certain functions of integration (appendix III), and since a formal Kirchhoff solution of (7·10) can be written down, the complete solution of equations (7·3) to (7·9) may be given, as in appendix III.

Equation (7·10) shows that if the functions of integration u, v and w are put equal to zero, α satisfies the wave equation in spherical polar co-ordinates. In accordance with the argument in §§ 3 and 4 we seek now solutions of the $(1s)$ approximations corresponding to the wave field of two symmetrically moving particles. As a basis for such solutions we may use for α expressions such as G_2 and G_4 in (3·9) and (3·10), both of which satisfy the wave equation. In analogy with electromagnetism I shall call the solution derived from G_2 the quadrupole wave solution.

With appropriate choice of functions of integration the quadrupole wave solution may be taken as (appendix III)

$$\left.\begin{aligned}
\overset{(11)}{A} &= -\tfrac{4}{3}r^{-1}(f^2)'' - P_2[\tfrac{8}{3}r^{-1}(f^2)'' + 8r^{-2}(f^2)' + 8r^{-3}f^2], \\
\overset{(11)}{B} &= \tfrac{4}{3}r^{-1}(f^2)'' + 4r^{-2}(f^2)' - 4r^{-1}\int_\infty^r r^{-3}f^2\,\mathrm{d}r \\
&\quad + P_2\left[-\tfrac{4}{3}r^{-1}(f^2)'' + 4r^{-3}f^2 + 4r^{-1}\int_\infty^r r^{-3}f^2\,\mathrm{d}r\right], \\
\overset{(11)}{C} &= -\tfrac{4}{3}r^{-1}(f^2)'' - \tfrac{4}{3}r^{-2}(f^2)' - \tfrac{8}{3}r^{-3}f^2 - \tfrac{4}{3}r^{-1}\int_\infty^r r^{-3}f^2\,\mathrm{d}r \\
&\quad + P_2\left[\tfrac{4}{3}r^{-1}(f^2)'' + \tfrac{16}{3}r^{-2}(f^2)' + \tfrac{20}{3}r^{-3}f^2 + \tfrac{4}{3}r^{-1}\int_\infty^r r^{-3}f^2\,\mathrm{d}r\right], \\
\overset{(11)}{D} &= -\tfrac{4}{3}r^{-1}(f^2)'' + \tfrac{8}{3}r^{-2}(f^2)' - 8r^{-3}f^2 - 32r\int_\infty^r r^{-5}f^2\,\mathrm{d}r \\
&\quad - P_2\left[\tfrac{8}{3}r^{-1}(f^2)'' + \tfrac{8}{3}r^{-2}(f^2)' + 8r^{-3}f^2 + 16r\int_\infty^r r^{-5}f^2\,\mathrm{d}r\right].
\end{aligned}\right\} \tag{7·11}$$

The function f is to be calculated throughout at time $t-r$, and $'$ means differentiation with respect to this argument.

As will be explained later we shall use only those terms in (7·11) which are of order r^{-1} for large r. These are

$$\left.\begin{aligned}
\overset{(11)}{A} &= r^{-1}h\cos^2\theta, \\
\overset{(11)}{B} &= -\tfrac{1}{2}r^{-1}h\sin^2\theta, \\
\overset{(11)}{C} &= \tfrac{1}{2}r^{-1}h\sin^2\theta, \\
\overset{(11)}{D} &= r^{-1}h\cos^2\theta,
\end{aligned}\right\} \tag{7·12}$$

where
$$h = -4\frac{\partial^2}{\partial t^2}\{[f(t-r)]^2\}. \tag{7·13}$$

The solution of the (12) approximation is given by an expression similar to (7·11), but more complicated and containing terms in $P_4(\cos\theta)$. It is not necessary to give this here as only terms in r^{-1} will be required. These are (appendix III)

$$\left.\begin{aligned}
\overset{(12)}{A} &= r^{-1}k\cos^4\theta, \\
\overset{(12)}{B} &= -r^{-1}k\sin^2\theta(1-\tfrac{2}{3}\sin^2\theta), \\
\overset{(12)}{C} &= r^{-1}k\sin^2\theta(1-\tfrac{2}{3}\sin^2\theta), \\
\overset{(12)}{D} &= r^{-1}k\cos^4\theta,
\end{aligned}\right\} \tag{7·14}$$

where
$$k = -\frac{1}{4}\frac{\partial^4}{\partial t^4}\{[f(t-r)]^4\}. \tag{7·15}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

It was pointed out in § 2 that the symmetry of our problem would be expected to exclude any dipole wave. We shall now verify that this is so. The dipole wave solution of the wave equation is proportional to

$$\alpha = (r^{-1}f' + r^{-2}f)\cos\theta. \tag{7·16}$$

Since this makes the left-hand side of (7·10) zero we must have

$$\int (v_{11} + r^{-1}v_1)\,\mathrm{d}\theta - w_1 - r^{-1}w - u_1 - r^{-1}u = 0,$$

which is a relation between three of the functions of integration. It is shown in appendix III that

$$\gamma = -\alpha + \operatorname{cosec}^2\theta \int \left[\sin\theta\cos\theta\left(2\alpha + 2r^{-1}\int \alpha\,\mathrm{d}r + r^{-1}\int ru\,\mathrm{d}r + r^{-1}d\right)\right]\mathrm{d}\theta$$
$$+ \operatorname{cosec}^2\theta\left\{\int v\sin^2\theta\,\mathrm{d}\theta + q\right\}, \tag{7·17}$$

where $d(\theta, t)$ and $q(r, t)$ are further functions of integration. Inserting (7.16) into (7.17) and carrying out some of the integrations, we have

$$\gamma = -\alpha - \tfrac{2}{3}\cot^2\theta\cos\theta\, r^{-1}f' + \operatorname{cosec}^2\theta\int v\sin^2\theta\,\mathrm{d}\theta + q\operatorname{cosec}^2\theta$$
$$+ \operatorname{cosec}^2\theta \int\left[\sin\theta\cos\theta\left(r^{-1}\int ru\,\mathrm{d}r + r^{-1}d\right)\right]\mathrm{d}\theta,$$

so that γ will be singular along the z-axis unless we can wipe out the term

$$-\tfrac{2}{3}\cot^2\theta\cos\theta\, r^{-1}f' \tag{7·18}$$

by appropriate choices of the functions of integration. Since f' is a function of $t-r$, $d(\theta, t)$ is useless for this purpose. (We exclude functions f which are polynomials in $t-r$.) Moreover, u and v are not functions of t, so (7·18) can be wiped out only by the use of the function $q(r, t)$. If, however, we were to choose

$$q = \text{const.} \times r^{-1}f',$$

there would still be left a singularity in γ at either $\theta = 0$ or $\theta = \pi$, or both.

Hence, it is not possible to choose the arbitrary functions so that γ is non-singular along the z-axis, and we conclude that there is no dipole wave for the metric given by (5·8).

8. Preliminaries to the (22) approximation

As stated in the Introduction, the aim of this paper is to find whether the energy which seems to be transmitted by gravitational waves is represented as a loss of gravitational mass, or as some other permanent change in the sources. A simple experiment to study this question would be the following. The two particles A and B of § 2 are at rest at the ends of a compressed spring until a certain time t_0 when a time switch operates to release the spring. The particles then oscillate sinusoidally until time t_1, when they are secured in their original position by some mechanism also operated by the switch. Between times t_0 and t_1 energy has presumably been lost as gravitational radiation. Is the static system after t_1 different from that before t_0?

One may, of course, point out that the system is not strictly static before t_0 because there must be motion in the mechanism operating the switch. Until we have some knowledge of the type of solutions offered by the field equations to this problem, we cannot know whether this objection is valid or not. For the present, the best procedure seems to be to ignore subtleties of this sort, and to assume that the field equations take no account of them.

From the mathematical point of view we must note that, although we may assume before t_0 a static metric with axial symmetry, the metric after t_1 will not be static because the gravitational waves will still be present somewhere in space. However, we should expect to obtain an answer to our question by examining the metric for space-time when t is very large, so that the waves are a long way off.

One difficulty in finding a solution to the problem in this form is that of satisfying boundary conditions at $t = t_0$ and $t = t_1$. Whilst the system is oscillating one can start the solution by taking a retarded potential such as (3·8). Then on the null surface $t-r = t_0$ one must match the solution to a static metric with axial symmetry, the conditions of matching being those given by O'Brien & Synge (1952), or by Lichnerowicz (1955). On the surface $t-r = t_1$ certain continuity conditions will need to be satisfied, though at this limit it may not be possible to match the solution to a static metric. The approximation method of this paper is not very suitable for dealing with these boundary-value problems, and it is preferable to circumvent them by a different approach.

We shall therefore choose for $f(t)$ a bounded function which has derivatives of all orders for $-\infty \leqslant t \leqslant \infty$. Examples are

$$f(t) = 2+\tanh \omega t \quad (\omega > 0), \tag{8·1}$$

and

$$f(t) = 1+(1+t^2)^{-1}. \tag{8·2}$$

In the former case the particle A would start at $t = -\infty$ from a distance $+1$ from the origin and finish at $t = +\infty$ at a distance $+3$ from the origin. Particle B would undergo a symmetrical motion on the negative z-axis. The motion (8.2) is similar, but the particles return to their starting points. With such functions the problem of boundary conditions does not arise.

If the emission of waves is accompanied by a loss of gravitational mass, the coefficient of $1/r$ in the g_{ik} at $t = +\infty$ should be different from that at $t = -\infty$. Since functions $f(t)$ of the type described have all their derivatives zero at $t = \pm\infty$, it is easy to see from the form of the $\overset{(1s)}{g}_{ik}$ that no such difference appears in the (1s) approximation. A difference *does* appear, however, in the (22) approximation, and it will be the object of the remaining sections to show this.

Since our main concern will be to find the difference in the g_{ik} at the times $t = \pm\infty$, we shall in the (22) approximation, dealt with in the next section, persistently disregard terms which tend to zero both when $t \to +\infty$ and when $t \to -\infty$. We shall also neglect terms which are of order $(1/r)^2$ or higher as $t \to \pm\infty$, whether or not these tend to zero as $t \to \pm\infty$. This is because the terms which represent loss of gravitational mass will be of order $1/r$ as $t \to \pm\infty$. By this means we shall obtain a solution which is valid for

$$0 \leqslant r \leqslant |t|,$$

or rather more precisely, valid as $t \to \pm\infty$, and for r which is less than $|t|$ but still great enough for $(1/r)^2$ to be negligible compared with $1/r$.

We shall not need to choose a definite function $f(t)$, though it may be helpful in §§ 9 and 10 to keep in mind the examples (8·1) and (8·2). The precise restrictions which we place on $f(t)$ are:

(i) $f(t)$ is single-valued and possesses derivatives of all orders for $-\infty \leqslant t \leqslant \infty$;

(ii) $f(t)$ tends to limits as $t \to \pm\infty$;

(iii) the derivatives of $[f(t)]^2$ tend to zero as $t \to \pm\infty$ at least as rapidly as t^{-1}.

In § 9 we shall meet the expression

$$X(\xi) \equiv \tfrac{1}{4}[h'(\xi)]^2 + \tfrac{1}{2}h(\xi)\,h''(\xi),$$

where h is the function given in (7·13). From (i) and (iii) it follows that $h(\xi)$ and $X(\xi)$, and all their derivatives, exist for $-\infty \leqslant \xi \leqslant \infty$, that $h(\xi)$ and its derivatives tend to zero at least as rapidly as ξ^{-1} as $\xi \to \pm\infty$, and that $X(\xi)$ and its derivatives tend to zero at least as rapidly as ξ^{-2} as $\zeta \to \pm\infty$.

9. The (22) approximation

Let us write the (22) approximation in the form

$$\Phi_{lm}\big(\overset{(22)}{g}_{ik}\big) = \Psi_{lm}\big(\overset{(11)}{g}_{ik}, \overset{(10)}{g}_{ik}, \overset{(12)}{g}_{ik}\big). \tag{9·1}$$

The left-hand side is linear in the $\overset{(22)}{g}_{ik}$ and their derivatives. The right-hand side contains terms of two types:

(i) products of the $\overset{(11)}{g}_{ik}$ and their derivatives;

(ii) products of $\overset{(10)}{g}_{ik}$ (and their derivatives) with $\overset{(12)}{g}_{ik}$ (and their derivatives).

All terms on the right-hand side are known from previous approximations.

The full equations (9·1) may be calculated as described in appendix II. If we introduce the notation of (6.3), they may be written

$$\nu_{11}+\sigma_{11}+\rho_{11}+2r^{-1}(\nu_1+\sigma_1-\mu_1)+r^{-2}(\mu_{22}+\mu_2\cot\theta)-\mu_{44} = P, \tag{9·2}$$

$$\nu_{11}+r^{-1}(3\nu_1-\mu_1+\sigma_1+\rho_1)+r^{-2}(2\nu-2\mu)-\nu_{44} + r^{-2}(\mu_{22}+\sigma_{22}+\rho_{22}-\nu_2\cot\theta+2\sigma_2\cot\theta) = Q, \tag{9·3}$$

$$\sigma_{11}+r^{-1}(3\sigma_1-\mu_1+\nu_1+\rho_1)+r^{-2}(2\nu-2\mu)-\sigma_{44}+r^{-2}\sigma_{22} + r^{-2}\cot\theta(2\sigma_2-\nu_2+\mu_2+\rho_2) = R, \tag{9·4}$$

$$\mu_{44}+\nu_{44}+\sigma_{44}-\rho_{11}-2r^{-1}\rho_1-r^{-2}(\rho_{22}+\rho_2\cot\theta) = S, \tag{9·5}$$

$$\sigma_{12}+\rho_{12}+(\sigma_1-\nu_1)\cot\theta-r^{-1}(\mu_2+\rho_2) = L, \tag{9·6}$$

$$\nu_{14}+\sigma_{14}+r^{-1}(\nu_4+\sigma_4-2\mu_4) = M, \tag{9·7}$$

$$\mu_{24}+\sigma_{24}+\cot\theta(\sigma_4-\nu_4) = N, \tag{9·8}$$

where P, Q, R, S, L, M, N stand for the expressions Ψ_{lm} in (9·1). It will be noted that the left-hand sides of these equations are formally similar to the left-hand sides of the equations of the (1s) approximation, (7·3) to (7·9).

We can achieve a formal integration of equations (9·2) to (9·8) by the method given in appendix III. First, (9·8) is integrated with respect to the time and then substituted into

(9·6) to find ρ_1. This, and the integral of (9·7) are then substituted into (9·2). The result is an inhomogeneous wave equation for μ:

$$\Box\mu \equiv \mu_{11} + 2r^{-1}\mu_1 + r^{-2}(\mu_{22} + \mu_2 \cot\theta) - \mu_{44}$$
$$= P - \int (L_1 + r^{-1}L)\,\mathrm{d}\theta - \int (M_1 + r^{-1}M)\,\mathrm{d}t + \int \left\{ \int (N_{11} + r^{-1}N_1)\,\mathrm{d}t \right\} \mathrm{d}\theta$$
$$+ \int (v_{11} + r^{-1}v_1)\,\mathrm{d}\theta - (u_1 + r^{-1}u) - (w_1 + r^{-1}w), \qquad (9\cdot9)$$

where $u(r, \theta)$, $v(r, \theta)$ and $w(r, t)$ are functions of integration. The integration process leading to (9·9) gives intermediate stages from which σ, ν and ρ may be written down in terms of integrals of μ, P, Q, R, S, L, M, N and various functions of integration (appendix III, equations (III. 6), (III. 7) and (III. 4)).

The key to the whole solution is the value of μ given by (9·9). It is clear that this value is indeterminate to the extent of a solution of the homogeneous wave equation

$$\Box\mu = 0.$$

In the following we shall take the view that the essential sources of the wave field have already been inserted in the solutions chosen for the (1*s*) approximations, and that no further source functions are to be used other than those necessary to satisfy the inhomogeneous equation (9·9).

The work needed to calculate P, Q, etc., from the full expressions for $\overset{(11)}{g_{ik}}$ (given by (7·11)) and for $\overset{(12)}{g_{ik}}$ is quite prohibitive, so we are driven to look for a method of finding the leading terms in μ, ν, σ and ρ without doing the full computation. As explained in § 8 we are interested in the difference in the metric at $t = +\infty$ from that at $t = -\infty$, and, in particular, we wish to find the differences in the terms in r^{-1} which might correspond to a loss of gravitational mass.

If we were to work out the right-hand side of (9·9) using the complete expressions for $\overset{(11)}{g_{ik}}$ and $\overset{(12)}{g_{ik}}$ we should obtain

$$\Box\mu = \sum_{n=2}^{8} r^{-n} \overset{(n)}{p}[(t-r), \theta] + J + \int (v_{11} + r^{-1}v_1)\,\mathrm{d}\theta - (u_1 + r^{-1}u) - (w_1 + r^{-1}w), \qquad (9\cdot10)$$

where $\overset{(n)}{p}$ would be known functions, and J would be a known expression involving integrals such as occur in (7·11), and occur also in the corresponding formulae for the $\overset{(12)}{g_{ik}}$. Remembering that we are interested only in the coefficient of $1/r$ in μ, it is fairly obvious at once that we may ignore J. To justify this, we notice that for a typical integral term in (7·11) we have, since f^2 is bounded (say, $f^2 < K$, where K is a positive constant)

$$\left| r^{-1} \int_\infty^r r^{-3} f^2 \,\mathrm{d}r \right| < K r^{-1} \left| \int_\infty^r r^{-3}\,\mathrm{d}r \right| < K r^{-3}. \qquad (9\cdot11)$$

All the integrals in (7·11) satisfy a similar inequality, and integral terms which arise in $\overset{(12)}{g_{ik}}$ are less than Kr^{-5}. Owing to the type of quadratic product which occurs in Ψ'_{lm}, it follows that terms in J must be of order $(1/r)^4$, or of higher order in $1/r$. They may therefore be expected to induce terms in μ of order $(1/r)^2$ or higher, and not to affect the $1/r$ term. We shall therefore neglect J in what follows.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Let us consider the contribution to μ necessitated by a term $r^{-n}\overset{(n)}{p}$ on the right of (9·10). For $n \geqslant 4$ we assume that there will be no contribution to the $1/r$ term in μ, in which we are interested. We shall deal later with the term $r^{-2}\overset{(2)}{p}$. For the present we shall try to construct a solution of

$$\Box\mu = r^{-3}\overset{(3)}{p}. \qquad (9{\cdot}12)$$

Now $\overset{(3)}{p}[(t-r),\theta]$ will consist of a series of members each of which is a product of a function of $t-r$ with a function of θ, say

$$\overset{(3)}{p} = \sum_i \overset{(i)}{l}(t-r)\,\overset{(i)}{s}(\theta).$$

We shall take one of these members and solve the equation

$$\Box\mu = r^{-3}\overset{(i)}{l}\,\overset{(i)}{s}. \qquad (9{\cdot}13)$$

It is easily verified that

$$\mu = r^{-1}\overset{(i)}{l}\Theta(\theta) \qquad (9{\cdot}14)$$

is a solution of (9·13) provided that

$$\Theta_{22} + \Theta_2 \cot\theta = \overset{(i)}{s}. \qquad (9{\cdot}15)$$

Now $\overset{(i)}{s}$ will have the form

$$\overset{(i)}{s} = k_1 \sin^4\theta + k_2 \sin^2\theta + k_3, \qquad (9{\cdot}16)$$

where k_1, k_2 and k_3 are known constants, and it will not in general be possible to find a function Θ which satisfies (9·15) and is non-singular for $0 \leqslant \theta \leqslant \pi$.

The difficulty can be overcome by using the arbitrary function w which occurs in (9·10). If we choose w so that

$$w_1 + r^{-1}w = K\overset{(i)}{l}r^{-3}, \qquad (9{\cdot}17)$$

where K is a constant, (9·13) becomes

$$\Box\mu = r^{-3}\overset{(i)}{l}(\overset{(i)}{s} - K),$$

and using again the form (9·14) for μ, (9·15) becomes

$$\Theta_{22} + \Theta_2 \cot\theta = \overset{(i)}{s} - K. \qquad (9{\cdot}18)$$

It is always possible to choose K so that this equation has a non-singular solution if $\overset{(i)}{s}$ has the form (9·16).

By this method we obtain a satisfactory solution of (9·13), and by repeating this process for each of the products $\overset{(i)}{l}\,\overset{(i)}{s}$ we can build up a solution of (9·12).

We now investigate the contribution to μ from the term $r^{-2}\overset{(2)}{p}$ in the series on the right of (9·10). As was the case with $r^{-3}\overset{(3)}{p}$, so is $r^{-2}\overset{(2)}{p}$ a sum of products of functions:

$$\overset{(2)}{p} = \sum_i \overset{(i)}{n}(t-r)\,\overset{(i)}{c}(\theta); \qquad (9{\cdot}19)$$

and as before, the solution of

$$\Box\mu = r^{-2}\overset{(2)}{p} \qquad (9{\cdot}20)$$

can be built up from solutions of

$$\Box\mu = r^{-2}\overset{(i)}{n}\,\overset{(i)}{c}. \qquad (9{\cdot}21)$$

Let us in (9·21) substitute

$$\mu = \mu^* - \tfrac{1}{2} r^{-1} \overset{(i)}{c} \ln r \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi;$$

the result is

$$\Box \mu^* = \tfrac{1}{2} r^{-3} \left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta\right) \ln r \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi - \tfrac{1}{2} r^{-3} \overset{(i)}{c} \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi. \tag{9·22}$$

Next, write

$$\mu^* = \mu^{**} + \tfrac{1}{4} r^{-2} \left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta\right) \ln r \int_{-\infty}^{t-r} \mathrm{d}\eta \int_{-\infty}^{\eta} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi,$$

and substitute this into (9·22). It is found that μ^{**} has to satisfy

$$\Box \mu^{**} = -\tfrac{1}{4} r^{-4} \ln r \left[\left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta\right)_{22} + \cot\theta \left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta\right)_2 + 2\left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta\right)\right] \int_{-\infty}^{t-r} \mathrm{d}\eta \int_{-\infty}^{\eta} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi$$
$$+ \tfrac{3}{4} r^{-4} \left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta\right) \int_{-\infty}^{t-r} \mathrm{d}\eta \int_{-\infty}^{\eta} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi + \tfrac{1}{2} r^{-3} \left(\overset{(i)}{c}_{22} + \overset{(i)}{c}_2 \cot\theta - \overset{(i)}{c}\right) \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi. \tag{9·23}$$

The first two terms on the right are of order r^{-4} and we assume that they give no contribution of order r^{-1} in μ^{**}. The contribution to μ^{**} from the last term on the right, which is of order r^{-3}, is given by an expression of the form (9·14), which was the solution of (9·13). Up to order r^{-1} therefore, the solution of (9·21) for μ has the form

$$\mu = -\tfrac{1}{2} r^{-1} \overset{(i)}{c} \ln r \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi + r^{-1} \overset{(i)}{c}{}^* \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, \mathrm{d}\xi, \tag{9·24}$$

where $\overset{(i)}{c}{}^*$ is a function of θ determined by the procedure which led to (9·18). This analysis assumes that $\overset{(i)}{n}$ may be integrated twice without introducing singularities; we shall later check that this assumption is justified. We may use this procedure on each of the terms in (9·19), thereby obtaining a solution of (9·20).

In this way, using the various contributions of types (9·14) and (9·24) we can build up the solution of (9·10) for μ as far as the term in r^{-1}. We do not need to use the arbitrary functions u and v, and may put them equal to zero. To ensure that our solution is non-singular, the function w may, however, be needed.

The next step is to consider the actual expression for the right-hand side of (9·10). Let us take for the $\overset{(1s)}{g}_{ik}$ the approximate expressions given by (7·12) and (7·14), and work out the right-hand sides of (9·2) to (9·8) using only these, and working only to order r^{-2}. We find (appendix II):

$$\left.\begin{aligned}
P &= r^{-2} X \sin^4\theta, \\
Q &= -8 r^{-2} k'' \sin^2\theta \left(1 - \tfrac{2}{3}\sin^2\theta\right), \\
R &= 8 r^{-2} k'' \sin^2\theta \left(1 - \tfrac{2}{3}\sin^2\theta\right), \\
S &= r^{-2} X \sin^4\theta, \\
L &= -\tfrac{1}{2} r^{-2} h h' \sin\theta\cos\theta (1 - 9\cos^2\theta) - 16 r^{-2} k' \cos^3\theta \sin\theta, \\
M &= -r^{-2} X \sin^4\theta, \\
N &= \tfrac{1}{2} r^{-2} h h' \sin\theta\cos\theta (1 - 9\cos^2\theta) - 16 r^{-2} k' \cos^3\theta \sin\theta,
\end{aligned}\right\} \tag{9·25}$$

where $'$ means $\partial/\partial t$, $k(t-r)$ is the function given by (7·15), and

$$X(t-r) = \tfrac{1}{4}(h'^2 + 2hh''). \tag{9·26}$$

The formulae (9·25) have to be substituted into (9·9) so that the latter may be written in the form (9·10). It is easy to see that if this is done, the right-hand side of (9·10) will be correct to order r^{-2}, but not to order r^{-3}. To achieve accuracy to order r^{-3} we should have to calculate P, Q, etc., to order r^{-3}. However, in spite of this, it turns out that the approximation (9·25) is sufficient for our purposes. The reason is as follows. We seek only those terms in r^{-1} in μ which exhibit a permanent change in the metric, that is, terms which have different values at $t=\pm\infty$. Suppose that such a term in μ arises from a term in r^{-3} on the right of (9·10). Now the presence of this term in μ means that the function $\overset{(i)}{l}$ in (9·14) has different values at $t=\pm\infty$ (r finite); hence *the terms in* r^{-3} *in* (9·10) *which interest us are only those whose coefficients have different values at* $t=\pm\infty$. It happens that such terms all arise from the expressions (9·25) and so there is no need to calculate P, Q, etc., to r^{-3}.

To prove this last statement, let us study the way in which terms in r^{-3} on the right of (9·10) arise. They come from P and from the integrated expressions in (9·9). Consider, for example, the function M, which may be written

$$M=\sum_{n=2}^{7} r^{-n}\overset{(n)}{M}{}^*[(t-r),\theta]+J^*,$$

J^* being an expression involving integrals the contribution of which to (9·9) we ignore for reasons previously given. The contribution of M to the right-hand side of (9·9) is

$$-\int(M_1+r^{-1}M)\,\mathrm{d}t=r^{-2}\overset{(2)}{M}{}^*+r^{-3}\int\overset{(2)}{M}{}^*\,\mathrm{d}t+r^{-3}\overset{(3)}{M}{}^*+O(r^{-4}). \tag{9·27}$$

Now from (9·25), $\overset{(2)}{M}{}^*=-X\sin^4\theta$, which tends to zero as $t-r\to\pm\infty$; and it is not hard to see that $\overset{(3)}{M}{}^*$ also tends to zero as $t-r\to\pm\infty$ because every term in it contains at least one derivative of f^2 as a factor. Consider, however, the following integral (r finite):

$$\begin{aligned} r^{-3}\int_{-\infty}^{\infty}\overset{(2)}{M}{}^*\,\mathrm{d}t &= -\tfrac{1}{4}r^{-3}\sin^4\theta\int_{-\infty}^{\infty}(h'^2+2hh'')\,\mathrm{d}t,\\ &= -\tfrac{1}{4}r^{-3}\sin^4\theta\left\{[2hh']_{-\infty}^{\infty}-\int_{-\infty}^{\infty}h'^2\,\mathrm{d}t\right\},\\ &= \tfrac{1}{4}r^{-3}\sin^4\theta\int_{-\infty}^{\infty}h'^2\,\mathrm{d}t; \end{aligned} \tag{9·28}$$

this is not zero, because h' is not identically zero. Thus of the terms written out on the right of (9·27), the second will give a permanent change in μ but the others will not. The omission of the term in r^{-3} in M in (9·25) means that $\overset{(3)}{M}{}^*$ in (9·27) is missing; but this does not matter because even if we had it, it would not lead to a permanent change in μ of order $1/r$.

In the same way we can deal with the contributions of the functions P, L and N to (9·9). In each case one finds that the coefficient of r^{-3} is zero at $t=+\infty$ and at $t=-\infty$. Hence, these contributions will not lead to any permanent change in μ (of order $1/r$) and may be ignored. Thus the only term in r^{-3} which interests us is that which comes from M, and which we take to be

$$-r^{-3}\sin^4\theta\int_{-\infty}^{t-r}X(\xi)\,\mathrm{d}\xi.$$

The coefficient $\overset{(2)}{p}$ in (9·10) may be calculated accurately from our expressions (9·25) and is

$$\overset{(2)}{p} = 8k'' \cos^4 \theta.$$

This will give a contribution to μ of the form (9·24), where

$$\int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, d\xi = \int_{-\infty}^{t-r} k''(\xi)\, d\xi = k'(t-r), \qquad (9·29)$$

since $k'(-\infty) = 0$. From the expression for k in (7·15) we see that (9·29) has the same value (i.e. zero) at $t = \pm\infty$ (r finite), so that the contribution to μ from (9·24) represents no permanent change in the metric. This too we may therefore ignore. We easily verify that

$$\int_{-\infty}^{t-r} d\eta \int_{-\infty}^{\eta} \overset{(i)}{n}(\xi)\, d\xi$$

is not singular, thus vindicating the hypothesis used in deriving (9·24).

In obtaining the partial solution (9·24) of (9·21) we invoked a procedure, described earlier in this section, which makes use of the function of integration w. This was when we obtained the last term on the right of (9·24) from the last term on the right of (9·23). In order that the function $\overset{(i)}{c^*}$ shall not be singular it is necessary to choose w to satisfy

$$w_1 + r^{-1} w = K r^{-3} \int_{-\infty}^{t-r} \overset{(i)}{n}(\xi)\, d\xi;$$

(see (9·17)). From this we find, ignoring a function of integration which leads to terms of no interest

$$w = r^{-1} \int_{\infty}^{r} \eta^{-2}\, d\eta \int_{-\infty}^{t-\eta} \overset{(i)}{n}(\xi)\, d\xi.$$

The function w enters the expression for ρ (appendix III, equation (III.4)), giving a contribution

$$r \int_{\infty}^{r} \chi^{-2}\, d\chi \int_{\infty}^{\chi} \eta^{-2}\, d\eta \int_{-\infty}^{t-\eta} \overset{(i)}{n}(\xi)\, d\xi. \qquad (9·30)$$

Performing two partial integrations, and bearing in mind (9·29), it is not difficult to show that (9·30) yields an expression which is of order r^{-2} and therefore of no interest to us here.

We are at last in a position to derive the solution for μ. The equation to be solved is

$$\Box \mu = -r^{-3} \sin^4 \theta \int_{-\infty}^{t-r} X(\xi)\, d\xi. \qquad (9·31)$$

From the long argument above it should be clear that this will give all the terms in μ, of order r^{-1}, which show permanent change. Equation (9·31) has the form of (9·13) and the solution, given by (9·14) is

$$\mu = r^{-1} \Theta(\theta) \int_{-\infty}^{t-r} X(\xi)\, d\xi, \qquad (9·32)$$

where Θ has to satisfy an equation of type (9·18), namely

$$\Theta_{22} + \Theta_2 \cot \theta = -\sin^4 \theta - K,$$

K being an arbitrary constant, to be chosen so that Θ is non-singular for $0 \leqslant \theta \leqslant \pi$. It is found that we must take

$$K = -\tfrac{8}{15},$$

in which case

$$\Theta = a_0 + \tfrac{2}{15} \sin^2 \theta + \tfrac{1}{20} \sin^4 \theta, \qquad (9·33)$$

where a_0 is an arbitrary constant.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

As previously explained, the introduction of this value of K requires that the function w satisfy

$$w_1+r^{-1}w=-\tfrac{8}{15}r^{-3}\int_{-\infty}^{t-r}X(\xi)\,\mathrm{d}\xi,$$

of which the solution may be taken as

$$w=-\tfrac{8}{15}r^{-1}\int_{\infty}^{r}\eta^{-2}\,\mathrm{d}\eta\int_{-\infty}^{t-\eta}X(\xi)\,\mathrm{d}\xi. \tag{9·34}$$

Thus the required solution up to r^{-1} in μ, retaining only those terms which differ at $t=\pm\infty$, is (9·32) with Θ given by (9·33); the choice of this solution requires that we give to w the value (9·34).

We have still to find the remaining $\overset{(22)}{g}_{ik}$, which are given by equations (III. 5), (III. 6) and (III. 7) of appendix III. To use these we insert the value of μ and of w, and put $u=v=0$. It is found that the expressions L and N which occur do not add any terms of order r^{-1} which show permanent change. M does produce such a term, but this arises only from $\overset{(2)}{M}{}^*$ given by (9·25). The complete solution of the (22) approximation up to terms in r^{-1} which represent a permanent change in the metric is thus found to be

$$\mu=\overset{(22)}{A}=r^{-1}(a_0+\tfrac{2}{15}\sin^2\theta+\tfrac{1}{20}\sin^4\theta)\int_{-\infty}^{t-r}X(\xi)\,\mathrm{d}\xi, \tag{9·35}$$

$$\nu=\overset{(22)}{B}=r^{-1}(\tfrac{1}{15}\sin^2\theta+\tfrac{1}{30}\sin^4\theta)\int_{-\infty}^{t-r}X(\xi)\,\mathrm{d}\xi +r^{-1}(a_0+\tfrac{1}{5}\sin^2\theta-\tfrac{3}{4}\sin^4\theta)\int_{\infty}^{r}\eta^{-1}\,\mathrm{d}\eta\int_{-\infty}^{t-\eta}X(\xi)\,\mathrm{d}\xi, \tag{9·36}$$

$$\sigma=\overset{(22)}{C}=-r^{-1}(\tfrac{1}{15}\sin^2\theta+\tfrac{1}{30}\sin^4\theta)\int_{-\infty}^{t-r}X(\xi)\mathrm{d}\xi +r^{-1}(a_0+\tfrac{1}{15}\sin^2\theta-\tfrac{3}{20}\sin^4\theta)\int_{\infty}^{r}\eta^{-1}\,\mathrm{d}\eta\int_{-\infty}^{t-\eta}X(\xi)\,\mathrm{d}\xi, \tag{9·37}$$

$$\rho=\overset{(22)}{D}=-r(a_0+\tfrac{2}{15}\sin^2\theta+\tfrac{1}{20}\sin^4\theta)\int_{\infty}^{r}\xi^{-2}X(t-\xi)\,\mathrm{d}\xi+r\int_{\infty}^{r}\eta^{-1}w(\eta,t)\,\mathrm{d}\eta. \tag{9·38}$$

In these, X is given by (9·26) and w by (9·34); a_0 is an arbitrary constant. These satisfy the equations of the (22) approximation, (9·2) to (9·8), in the following sense:

(i) all terms of order r^{-1} vanish on both sides;

(ii) equations (9·2), (9·5) and (9·7) are satisfied up to order r^{-2};

(iii) equations (9·3), (9·4), (9·6) and (9·8) are satisfied up to order r^{-2} except for the terms in Q, R, L and N in (9·25), which do not give rise to a permanent change in the metric;

(iv) all integral expressions disappear when (9·35) to (9·38) are substituted into the left-hand sides of (9·2) to (9·8).

The discrepancy (iii) is simply a consequence of our persistent neglect of terms which led to no permanent change in μ, ν, σ and ρ, and it may be rectified if appropriate terms are retained during the process of solution. The additions to μ, ν, σ and ρ which thereby result are of no interest to us here. The point of (iv) is that the integrals in (9·35) to (9·38) have different values at $t=\pm\infty$, whereas the right-hand sides of (9·2) to (9·8) certainly do not (up to the order considered). It is, therefore, necessary to verify that the integrals disappear on substitution into the field equations.

During the course of verifying that the solution satisfies the field equations in the sense described above, it is necessary to differentiate the double integrals in (9·36) and (9·37) with respect to t, through the sign of integration. This is justified because the integrals are uniformly convergent for all finite values of t provided that $r > 0$ (see also next section).

It may be as well to summarize briefly this long and complicated procedure for deriving the solution (9·35) to (9·38) of the (22) approximation. From the field equations (9·2) to (9·8) we deduced the inhomogeneous wave equation (9·9) for $\mu\ (= \overset{(22)}{A})$, which is the key to the solution. Our procedure then was to pick out terms on the right of (9·9) which led to contributions to μ which (a) were of order r^{-1}, and (b) had different values at $t = \pm\infty$. All other contributions to μ were neglected. Once these terms were discovered, there was little difficulty in finding their contributions to μ: the only subtlety was that to avoid singularities in the angular part of such contributions it was necessary to use the function of integration $w(r, t)$. In this way we arrived at the expression (9·35) for $\overset{(22)}{A}$. The remaining coefficients $\overset{(22)}{B}$, $\overset{(22)}{C}$ and $\overset{(22)}{D}$ of the (22) approximation are given in terms of μ and other known expressions by equations in appendix III, and in working these out, we retained only those terms which are of order r^{-1} and which have different values at $t = \pm\infty$.

10. Examination of the solution of the (22) approximation

We shall now examine the solution (9·35) to (9·38), considering in the first place the integral occurring in (9·35). From (9·26) we have

$$\int_{-\infty}^{t-r} X(\xi)\,d\xi = [\tfrac{1}{2}h'h]_{-\infty}^{t-r} - \frac{1}{4}\int_{-\infty}^{t-r} h'^2\,d\xi; \tag{10·1}$$

recalling the properties of h assumed at the end of § 8, we see that the integral on the left exists for all t and r, and that

$$\int_{-\infty}^{\infty} X(\xi)\,d\xi = -\frac{1}{4}\int_{-\infty}^{\infty} h'^2\,d\xi.$$

Let us consider now another integral occurring in (9·36) and (9·37)

$$\int_{\infty}^{r} \eta^{-1}\,d\eta \int_{-\infty}^{t-\eta} X(\xi)\,d\xi. \tag{10·2}$$

It is not hard to see that, if t is finite and $r > 0$, this integral exists because for sufficiently large η,

$$\eta^{-1}\int_{-\infty}^{t-\eta} X(\xi)\,d\xi \sim \eta^{-2};$$

we have used here the assumption about X made at the end of § 8. In fact, for any finite range of t, the integral (10·2) is uniformly convergent, which justifies differentiation through the sign of integration, used in verifying the solution.

It turns out, however, that (10·2) tends to infinity at $t \to +\infty$. This may be proved rigorously, or may be seen in a qualitative way by noting that

$$\int_{-\infty}^{\xi_1} X(\xi)\,d\xi \sim \text{const.} \quad \text{if} \quad \xi_1 > \xi',$$

or $$\sim 0 \quad \text{if} \quad \xi_1 < \xi'',$$

where ξ' and ξ'' are certain constants ($\xi' > \xi''$). (For purposes of illustration one may think of $X(\xi)$ as of the form $(1+\xi^2)^{-n}$ $(n>1)$.) The effect of this for large t is to give (10·2) the approximate value

$$\text{const.} \times \int_{t-\xi'}^{r} \eta^{-1}\,d\eta$$

which tends to infinity as $t \to +\infty$.

Since we are particularly interested in the solution when $t \to \pm\infty$, this defect, unless it can be removed, completely vitiates the solution. *On calculating the components of the Riemann–Christoffel tensor one finds, however, that integrals of the form* (10·2) *are absent*. This suggests that the singularity has no physical significance, and in fact it can be removed by the transformation

$$\left.\begin{aligned}
r &= r^* + m^2a^4[(\tfrac{2}{15}-\tfrac{1}{2}a_0) - \tfrac{1}{15}\sin^2\theta^* - \tfrac{1}{40}\sin^4\theta^*] \int_\infty^{r^*} \eta^{-1}\,d\eta \int_{-\infty}^{t^*-\eta} X(\xi)\,d\xi \\
&\quad + m^2a^4[\tfrac{4}{45} - \tfrac{1}{9}\sin^2\theta^* - \tfrac{1}{36}\sin^4\theta^*]\, r^* \int_\infty^{r^*} \eta^{-2}\,d\eta \int_{-\infty}^{t^*-\eta} X(\xi)\,d\xi, \\
\theta &= \theta^* - m^2a^4(\tfrac{4}{45} + \tfrac{1}{90}\sin^2\theta^*)\sin\theta^*\cos\theta^* \int_\infty^{r^*} \eta^{-2}\,d\eta \int_{-\infty}^{t^*-\eta} X(\xi)\,d\xi \\
&\quad - m^2a^4(\tfrac{2}{15} + \tfrac{1}{10}\sin^2\theta^*)\sin\theta^*\cos\theta^*\, r^{*-1} \int_\infty^{r^*} \eta^{-1}\,d\eta \int_{-\infty}^{t^*-\eta} X(\xi)\,d\xi, \\
\phi &= \phi^*, \\
t &= t^* + m^2a^4(\tfrac{2}{15} - \tfrac{1}{2}a_0) \int_\infty^{r^*} d\eta \int_\infty^{\eta} \xi^{-1} X(t^*-\xi)\,d\xi.
\end{aligned}\right\} \quad (10\cdot3)$$

This has no effect on the (00), (1s) or (21) approximations, but it transforms the (22) approximation (i.e. the solution (9·35) to (9·38)) to

$$\left.\begin{aligned}
\overset{(22)}{g^*_{11}} &= -\tfrac{4}{15} r^{*-1} \int_{-\infty}^{t^*-r^*} X(\xi)\,d\xi + (\tfrac{8}{45} - \tfrac{2}{9}\sin^2\theta^* - \tfrac{1}{18}\sin^4\theta^*) \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta, \\
\overset{(22)}{g^*_{22}} &= r^{*2}(\tfrac{1}{15}\sin^2\theta^* + \tfrac{1}{30}\sin^4\theta^*) \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta, \\
\overset{(22)}{g^*_{33}} &= -r^{*2}\sin^2\theta^*\,(\tfrac{1}{15}\sin^2\theta^* + \tfrac{1}{30}\sin^4\theta^*) \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta, \\
\overset{(22)}{g^*_{44}} &= -\tfrac{4}{15} r^{*-1} \int_{-\infty}^{t^*-r^*} X(\xi)\,d\xi - \tfrac{8}{15} \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta \\
&\quad - (\tfrac{2}{15}\sin^2\theta^* + \tfrac{1}{20}\sin^4\theta^*)\, r^* \int_\infty^{r^*} \eta^{-2} X(t^*-\eta)\,d\eta, \\
\overset{(22)}{g^*_{12}} &= -(\tfrac{2}{9} + \tfrac{1}{9}\sin^2\theta^*)\, r^*\sin\theta^*\cos\theta^* \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta, \\
\overset{(22)}{g^*_{14}} &= (\tfrac{1}{15}\sin^2\theta^* + \tfrac{1}{40}\sin^4\theta^*) \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta \\
&\quad - (\tfrac{4}{45} - \tfrac{1}{9}\sin^2\theta^* - \tfrac{1}{36}\sin^4\theta^*)\, r^* \int_\infty^{r^*} \eta^{-2} X(t^*-\eta)\,d\eta, \\
\overset{(22)}{g^*_{24}} &= (\tfrac{2}{15} + \tfrac{1}{10}\sin^2\theta^*)\, r^*\sin\theta^*\cos\theta^* \int_\infty^{r^*} \eta^{-1} X(t^*-\eta)\,d\eta \\
&\quad + (\tfrac{4}{45} + \tfrac{1}{90}\sin^2\theta^*)\, r^{*2}\sin\theta^*\cos\theta^* \int_\infty^{r^*} \eta^{-2} X(t^*-\eta)\,d\eta.
\end{aligned}\right\} \quad (10\cdot4)$$

In obtaining these expressions it is necessary to differentiate some of the integrals in (10·3) through the integral sign. All these except (10·2) are uniformly convergent for $-\infty \leqslant t^* \leqslant \infty$; I shall assume that even in the case of (10·2) differentiation through the sign of integration is permissible for all t^*.

It will be noted that the $\overset{(22)}{g^*_{ik}}$ in (10·4) have non-diagonal form, like the metric (5·1). Before considering (10·4) in detail let us study the integrals occurring in it. Those of type (10·1) have already been dealt with, and we now turn to

$$\int_{\infty}^{r^*} \eta^{-1} X(t^*-\eta)\, \mathrm{d}\eta, \quad r^* \int_{\infty}^{r^*} \eta^{-2} X(t^*-\eta)\, \mathrm{d}\eta. \tag{10·5}$$

Remembering that $X(\xi) \to 0$ when $\xi \to \pm\infty$ at least as rapidly as ξ^{-2}, we easily see that (10·5) exist when t^* is finite ($r^* > 0$), and then tend to zero as $r^* \to \infty$ at least as rapidly as r^{*-2}. As $t^* \to \pm\infty$ the expressions (10·5) tend to zero at least as rapidly as t^{*-1} and t^{*-2}, respectively, as may be proved rigorously or seen qualitatively by thinking of $X(t^*-\eta)$ as a δ-function. If t^* and r^* tend to infinity in such a way that $t^*-r^* \to$ const. then (10·5) tend to zero by a similar argument.

It follows that the metric (10·4) has no singularity for $r^* > 0$. Moreover, owing to the fact that the derivatives of (10·5) tend to zero as r^* and t^* tend to infinity at least as rapidly as the expressions themselves, one can check that (10·4) tends to flatness as $r^* \to \infty$, $t^* \to \pm\infty$. (This is clearer on transforming (10·4) to Cartesian co-ordinates, because the contributions to the Cartesian metric from g^*_{22} and g^*_{33} in (10·4) are divided by r^{*2}, and those from g^*_{12} and g^*_{24} by r^*.) Therefore, (10·4) *represents an approximate space-time which is non-singular* (*except at* $r^* = 0$) *and which tends to flatness as* t^* *and* r^* *tend to infinity.*

Since the main results of this work will follow from the metric (10·4) it will perhaps be well to consider a little further the transformation by which we got it. Since it contains the integral (10·2) the transformation (10·3) is singular at $t^* = +\infty$, and the question arises whether it is permissible to remove singularities in the metric by transformations which are themselves singular in the same region. I think it would commonly be agreed that, until a proper definition of a physical singularity in the theory of relativity is given, this procedure is unavoidable: it is certainly used and the results obtained from it are considered significant, as in the case of Lemaître's work on the Schwarzschild singularity at $r = 2m$. In any case, the metric (10·4) is an approximate solution of the field equations in a sense which could be made precise (as was done for (9·35) to (9·38)), and the manner in which it is obtained is largely irrelevant. Unless one is prepared to say that a non-singular solution has to be examined with a view to *inducing* singularities in it by co-ordinate transformations (which at the present stage of our knowledge would seem like looking for trouble), the fact that one obtains, in one way or another, a non-singular solution should be sufficient.

Of course it may be that the co-ordinate system needed to ensure regularity of the (22) approximation will make the higher approximations singular. This could only be decided by pushing the solution to a higher stage of approximation: but the objection here seems no stronger than that which can be made against any approximation method of this type—that one cannot be sure that it will be possible to carry on with it indefinitely.

With this brief justification of the use of the transformation (10·3), I shall now proceed with the examination of the solution in the form (10·4). This examination will lead to results so plausible that they themselves offer some pragmatic justification for the procedure adopted.

In the first place it will be seen that the transformation (10·3) removes the arbitrary constant a_0, which may therefore be taken to have no physical significance. We are interested in the (22) approximation with

$$r \text{ finite}, \quad t \to \pm\infty,$$

and we are concerned only with terms in the metric which are of order r^{-1} and non-zero at $t = \pm\infty$; we therefore ignore expressions (10·5). The terms of interest in (10·4) which then remain are (omitting the asterisks):

$$\left.\begin{aligned} \overset{(22)}{g}_{11} &= -\tfrac{4}{15} r^{-1} \int_{-\infty}^{t-r} X(\xi)\,\mathrm{d}\xi, \\ \overset{(22)}{g}_{44} &= -\tfrac{4}{15} r^{-1} \int_{-\infty}^{t-r} X(\xi)\,\mathrm{d}\xi. \end{aligned}\right\} \tag{10·6}$$

These formulae correspond to an approximate Schwarzschild solution (terms of order r^{-2} being ignored) in which the mass is ΔM, given by

$$\Delta M = \tfrac{2}{15} m^2 a^4 \int_{-\infty}^{t-r} X(\xi)\,\mathrm{d}\xi.$$

For fixed r this depends on the time, and for $t = +\infty$ it becomes, using (9·26) and integrating by parts as in (9·28),

$$-\Delta M = \frac{m^2 a^4}{30} \int_{-\infty}^{\infty} [h'(t)]^2\,\mathrm{d}t. \tag{10·7}$$

Thus the solution of the (22) *approximation shows that the gravitational mass of the system at* $t = +\infty$ *is less than that at* $t = -\infty$ *by the amount* (10·7).

The interpretation of this result is that the sources lose mass because energy is lost as gravitational waves. This latter energy may be calculated by using the energy pseudo-tensor $\mathfrak{t}_{ik}$. If Cartesian co-ordinates are used, the (22) approximation to $\mathfrak{t}_{ik}$ is made up entirely of terms from the ($1s$) approximations ($s = 0, 1, 2$), which shows that the energy lost comes from the (11) and (12) waves. It turns out (see next paragraph) that the energy lost is precisely equal to (10·7), so that *the loss of gravitational mass is exactly accounted for by the energy transmitted as gravitational waves.*

If we substitute for $h(t)$ from (7·13) into (10·7) we find

$$-\Delta M = \frac{2}{15} \int_{-\infty}^{\infty} \left\{\frac{\mathrm{d}^3 I}{\mathrm{d}t^3}\right\}^2 \mathrm{d}t, \tag{10·8}$$

where I is the moment of inertia of the two particles about the plane $z = 0$. This is essentially the standard formula (calculated by using $\mathfrak{t}_{ik}$) for the loss of energy due to the waves (Landau & Lifshitz 1951, p. 331).

The result (10·8) does not tell us whether the loss in mass occurs in the particles or in the machine which drives them. In fact, the latter has not appeared in the calculations though we could have added a term corresponding to its mass in the (10) approximation; if we had done so the formula (10·8) would have been unaltered. Another interesting question which is not clear from (10·8) is whether the loss of mass would occur at all in the absence of a machine: that is to say, if the motion is purely gravitational. This will be discussed in § 12.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

11. The sources of the field†

If one takes any *approximate* solution of (1·1), one can, by substituting it into the equations

$$R^i_k - \tfrac{1}{2}\delta^i_k R = -8\pi T^i_k, \tag{11·1}$$

obtain an expression for the energy tensor, T^i_k, which corresponds to it. For an exact solution T^i_k vanishes at all non-singular points; at singularities T^i_k appears usually to be singular, and it is sometimes written in terms of δ-functions.

For an inexact solution T^i_k given by (11·1) will generally not be zero anywhere at all. A distribution of matter and stresses will appear throughout space-time, and one may regard this distribution, together with the singularities in T^i_k, as representing the 'sources' of the approximate solution. Of course, in the actual physical situation to which the solution is supposed to approximate, the sources will be different; in particular the continuous distribution will not be present. The difference will be a measure of the agreement between the approximate solution and the exact solution which one would really like to get.

If one is correctly approximating to the exact solution with the desired sources, one may hope that by repeated application of the method one could approach consistently nearer to the exact solution: this would show up by the continuous part of T^i_k becoming progressively smaller away from the singularities. One would also expect the value of T^i_k near the singularities to approximate in some way to the (singular) values which it takes in the corresponding exact solution; however, as we do not properly understand the mathematical character of the singularities of equations (1·1), the information to be gained in this way is limited.

As an illustration we may take the following approximation to the Schwarzschild solution:

$$ds^2 = -(1+2m/r)\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + (1-2m/r)\,dt^2. \tag{11·2}$$

Using (11·1) we find for the non-zero components of T^i_k

$$\left.\begin{aligned} 8\pi T^1_1 &= -\frac{4m^2}{r^4}\left(1-\frac{4m^2}{r^2}\right)^{-1}, \\ 8\pi T^2_2 = 8\pi T^3_3 &= \frac{4m^2}{r^4}\left(1-\frac{m}{r}\right)\left(1-\frac{4m^2}{r^2}\right)^{-2}, \\ 8\pi T^4_4 &= \frac{4m^2}{r^4}\left(1+\frac{2m}{r}\right)^{-2}. \end{aligned}\right\} \tag{11·3}$$

We notice that these values of T^i_k refer to a tenuous static distribution of matter and stresses throughout space-time, and to singularities at $r = 0$ and $r = 2m$. (The metric (11·2) has the wrong signature for $r < 2m$.) We could guess from this that (11·2) approximates to the exact solution for a static, spherically symmetric concentration of matter near the origin, surrounded by empty space.

It may be that the approximate solution differs in some essential way from the desired exact solution. A likely cause of this is that the sources of the two solutions are significantly different, and it is possible that an examination of T^i_k for the approximate solution may reveal the discrepancy.

† I am grateful to a Referee whose suggestions led to the introduction of this section.

As an example of this corrective use of T^i_k let us take the linear approximation to the non-static field with axial symmetry, considered in § 7. Suppose that we represent the field of the two moving particles by

$$\left.\begin{aligned} g_{11} &= -1 - ma^2\overset{(11)}{A}, \\ g_{22} &= -r^2\left(1 + ma^2\overset{(11)}{B}\right), \\ g_{33} &= -r^2\sin^2\theta\left(1 + ma^2\overset{(11)}{C}\right), \\ g_{44} &= 1 + ma^2\overset{(11)}{D}, \end{aligned}\right\} \tag{11·4}$$

where $\overset{(11)}{A}$, etc., are given by (7·11). By a calculation we find

$$T^{14} = \frac{1}{16\pi} m^2a^4[r^{-2}X(t-r)\sin^4\theta + O(r^{-3})] + O(m^3a^6).$$

Integrating T^{14} over a sphere of radius r, we find for the rate at which matter flows out

$$\tfrac{2}{15}m^2a^4X(t-r) + \ldots.$$

We may integrate this with respect to t between the limits $-\infty$ and $+\infty$ to obtain for the total matter which flows *into* an infinite sphere

$$\frac{m^2a^4}{30}\int_{-\infty}^{\infty}[h'(t)]^2\,dt + O(m^3a^6). \tag{11·5}$$

This means that in order to maintain the field (11·4) we should have to infuse an amount (11·5) of matter. Thus even without considering the energy pseudo-tensor we conclude that our approximate solution (11·4) cannot be considered at its face-value as representing two moving particles of constant mass: if the mass is to remain constant we must introduce more matter. Since in the actual physical system under investigation more matter is certainly *not* introduced, this is another way of saying that energy is lost as radiation, and in the exact solution to which we are approximating the masses must diminish. This is exactly the result found in §10 by taking the solution to a higher degree of approximation. Thus the correct interpretation, as will be verified below, is that the masses diminish because energy is carried away by gravitational waves.

Let us try to use this method to study the approximate solution obtained in this paper. Neglecting the terms in ma^4, which do not contribute to the main results of § 10, we may take this as

$$\left.\begin{aligned} g_{11} &= -1 - 4mr^{-1} - ma^2\overset{(11)}{A} + m^2a^4\overset{(22)}{g}_{11}, \\ g_{22} &= -r^2\left(1 + ma^2\overset{(11)}{B}\right) + m^2a^4\overset{(22)}{g}_{22}, \\ g_{33} &= -r^2\sin^2\theta\left(1 + ma^2\overset{(11)}{C}\right) + m^2a^4\overset{(22)}{g}_{33}, \\ g_{44} &= 1 - 4mr^{-1} + ma^2\overset{(11)}{D} + m^2a^4\overset{(22)}{g}_{44}, \\ g_{ik} &= m^2a^4\overset{(22)}{g}_{ik} \quad (i \neq k), \end{aligned}\right\} \tag{11·6}$$

where $\overset{(11)}{A}, \ldots, \overset{(11)}{D}$ are given by (7·11), and $\overset{(22)}{g}_{ik}$ are given by (10·4). Unfortunately, the expressions for T^i_k are extremely complicated and to get them into a surveyable form it is almost essential to expand in powers of m and a^2 as in the original treatment.

Two points can be made from the general form of the T^i_k calculated from (11·6), before expansion. First, it is not hard to show that if $t \to \pm\infty$ and $r \ll |t|$, T^i_k tends to the values (11·3), but with

$$2m \text{ in place of } m \quad \text{as} \quad t \to -\infty,$$

and

$$2m - \Delta M \text{ in place of } m \quad \text{as} \quad t \to +\infty.$$

This confirms that the sources of the approximate solution lose mass, as stated in § 10. Secondly, one can show that for no value of t is there a conical two-dimensional surface $\theta = \text{const.}$ on which T^i_k is singular. This includes the degenerate cases $\theta = 0$ and $\theta = \pi$, and precludes the possibility that energy or matter is piped in or out along the axis of symmetry.

To make further use of T^i_k it seems to be necessary to expand it in powers of m and a^2. For all except the lowest approximations the calculations become prohibitively heavy if one tries to go beyond the term in r^{-2}. The following are the $\overset{(ps)}{T^i_k}$ for $p, s \leqslant 2$:

$$\left.\begin{array}{l}
\overset{(00)}{T^i_k} = \overset{(10)}{T^i_k} = \overset{(11)}{T^i_k} = 0; \\
\overset{(20)}{T^i_k} = O(r^{-4}); \\
\overset{(21)}{T^i_k} = O(r^{-3}); \\
\qquad \text{except} \left\{\begin{array}{l}
8\pi \overset{(21)}{T^2_2} = 2m^2a^2h''r^{-2}\sin^2\theta + O(r^{-3}), \\
8\pi \overset{(21)}{T^3_3} = -2m^2a^2h''r^{-2}\sin^2\theta + O(r^{-3}), \\
8\pi \overset{(21)}{T^1_2} = 4m^2a^2h'r^{-2}\sin\theta\cos\theta + O(r^{-3}), \\
8\pi \overset{(21)}{T^4_2} = -4m^2a^2h'r^{-2}\sin\theta\cos\theta + O(r^{-3});
\end{array}\right. \\
\overset{(22)}{T^i_k} = O(r^{-3}) \\
\qquad \text{except} \left\{\begin{array}{l}
8\pi \overset{(22)}{T^1_2} = \tfrac{1}{4}m^2a^4h'hr^{-2}\sin\theta\cos\theta(1 - 9\cos^2\theta) + O(r^{-3}), \\
8\pi \overset{(22)}{T^4_2} = \tfrac{1}{4}m^2a^4hh'r^{-2}\sin\theta\cos\theta(1 - 9\cos^2\theta) + O(r^{-3}).
\end{array}\right.
\end{array}\right\} \quad (11·7)$$

Though this information is meagre, a few conclusions can be drawn from it. The vanishing of $\overset{(11)}{T^i_k}$ for $(r > 0)$ shows that in the (11) approximation there is no distributed energy; this is reassuring because it means that in this approximation, which was the essential starting point for the investigation, the source may be taken simply as the quadrupole corresponding to the two particles. The $\overset{(20)}{T^i_k}$ refer to a Schwarzschild solution of mass $2m$, as in (11·3).

The main point to be made from the rest of (11·7) is that the integrals

$$\int_{-\infty}^{\infty} \mathrm{d}t \int_S \overset{(21)}{T^{14}} \mathrm{d}S \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}t \int_S \overset{(22)}{T^{14}} \mathrm{d}S,$$

where S is an infinite sphere, centre the origin, are zero. Thus the approximate metric (11·6), unlike (11·4), does not need a permanent infusion of matter to maintain it, at any rate as far as the (22) approximation. The reason is that in (11·6) the energy lost as waves is accounted for by the diminishing mass of the mechanical system, and $\overset{(22)}{T^i_k}$ is not called upon to make up any defficiency. This shows that (11·6) is a more satisfactory solution than (11·4), because in the physical system to which we are approximating there is certainly not the flux of matter which T^{14} represents.

It is likely that if we used (11·6) to calculate

$$\int_{-\infty}^{\infty} \mathrm{d}t \int_S \overset{(22)}{T}{}^{14}\,\mathrm{d}S$$

we should find a permanent flux of matter through the infinite sphere. However, following our experience with the two solutions (11·4) and (11·6) we should expect to be able to add to (11·6) certain terms of order m^4a^8 which would give a new solution without permanent flux.

Thus, even though it has not been possible to proceed very far with the calculations, we can say that consideration of T^i_k for the approximate solution (11·6) gives us confidence that we are on the right track; at any rate there is no indication that the sources of the approximate solution are incompatible with those of our physical model.

12. Types of motion of the sources

One matter concerning the motion of the sources needs to be made clear at this point. The approximation method of E. I. & H. yields equations of motion of the particles present. This happens because care is taken to introduce no singularities other than poles corresponding to point masses; this ensures that the motion is purely gravitational and as such it is determined by the field equations.

The procedure here is different. Singularities corresponding to point masses were introduced into the (11) approximation, but no steps were taken to eliminate singularities of other types in that approximation, or in the higher ones. The method of multipole expansion which we have used is not a convenient one for studying singularities. If the motion of the particles is arbitrary (as we have supposed) singularities, such as the stress singularity of Bach & Weyl (1922), must be present, since these are necessary to prevent the particles from executing a purely gravitational motion. They correspond to the springs, etc., which run the motion. If we had an exact solution corresponding to our problem we should expect the nature of the singularities to become clear, as it is in the solution of Bach & Weyl for two particles at rest. It is the presence of the singularities, though hidden by the method of approximation, which allows us to suppose the motion to be arbitrary.

The restrictions on the motion (i.e. on $f(t)$) given in § 8 were imposed so that the integrals in the solution (e.g. (10·1)) converged. If one takes purely sinusoidal motion,

$$f = \sin \omega t,$$

the integral (10·1) tends to infinity with t (like t). The obvious explanation of this is that in the course of such motion an infinite amount of energy is lost.

It will be noted that the restrictions of § 8 on $f(t)$ require it to tend to limits as $t \rightarrow \pm\infty$. They do not require, however, that these limits shall be equal. The system loses mass whether or not the particles return to their starting point.

Simple examples of functions satisfying the conditions have already been given in (8·1) and (8·2). One can, of course, impose vibrations on these; for example, one could take instead of (8·1),

$$f(t) = 2 + \tanh \omega t \sin \omega^* t.$$

Such a function is slightly more realistic as a model of a vibrating system, but it still has the disadvantage that the motion takes an infinite time.

As was explained in § 8, the experiment in which the two particles execute some motion between finite times t_0 and t_1, and are at rest before and afterwards poses, in general, an initial (and final) value problem, the solution of which would be difficult using the method of this paper. I am indebted to Professor H. Bondi for the following suggestion which remedies this situation to a considerable extent. Consider the function

$$\left.\begin{aligned} f(t) &= 0 \quad (-\infty \leqslant t \leqslant 0), \\ f(t) &= \exp\{-k_1 t^{-2} - k_2(t-1)^{-2}\} \quad (0 \leqslant t \leqslant 1),\ (k_1, k_2 > 0), \\ f(t) &= 0 \quad (1 \leqslant t \leqslant \infty). \end{aligned}\right\} \tag{12·1}$$

This function satisfies all the requirements of § 8; in particular it has derivatives of all orders at $t = 0$ and $t = 1$. It represents a motion which takes place in a finite time, and if it is made vibratory by multiplying it by a sinusoidal factor it forms a good model for the experiment referred to in § 8. Our conclusions of § 10 will therefore apply to this experiment: mass would be lost in an amount equal to the energy radiated.

The interesting feature of $f(t)$ in (12·1) is the smooth transition from rest to motion at $t = 0$, and from motion to rest at $t = 1$. Let us now suppose that, by the use of functions of this or of another type, we can arrange a smooth transition between rest and an arbitrary motion which starts from the position of rest. Suppose, for example, that we can arrange smooth transitions at $t = 0$ and $t = t_1$ for the functions

$$\left.\begin{aligned} f_1(t) &= 0 \qquad (-\infty \leqslant t \leqslant 0), \\ f_2(t) &= \phi(t) \quad (0 \leqslant t \leqslant t_1), \\ f_3(t) &= \phi(t_1) \quad (t_1 \leqslant t \leqslant \infty), \end{aligned}\right\} \tag{12·2}$$

where t_1 is constant, $\phi(0) = 0$ and where $\phi(t)$ has derivatives of all orders for $0 \leqslant t \leqslant t_1$. This corresponds to motion according to $\phi(t)$ for $0 \leqslant t \leqslant t_1$, and rest before and afterwards, and as a result of our assumption about the smooth transitions, the conclusions of § 10 will apply.

For the motion (12·2) we have $h(t) = -4\mathrm{d}^2\{[\phi(t)]^2\}/\mathrm{d}t^2$ for $0 \leqslant t \leqslant t_1$, and $h(t) = 0$ before and after. The loss of mass in this case is, from (10·7),

$$-\Delta M = \frac{8m^2a^4}{15}\int_0^{t_1}\left\{\frac{\mathrm{d}^3}{\mathrm{d}t^3}[\phi(t)]^2\right\}^2 \mathrm{d}t,$$

neglecting the effect on ΔM of the short transition periods at $t = 0$ and $t = t_1$. For this loss to be zero we must have

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}[\phi(t)]^2 = 0$$

of which the solution is

$$\phi(t) = (k_1 t^2 + k_2 t + k_3)^{\frac{1}{2}}, \tag{12·3}$$

k_1, k_2 and k_3 being arbitrary constants. *This is the most general rectilinear motion for which no energy is lost*, at any rate up to the (22) approximation.

We turn now to the question whether energy is lost when the particles move freely under their own gravitation for $0 \leqslant t \leqslant t_1$. It will be noted that motion under the inverse square law of attraction, calculated by ordinary Newtonian dynamics, is not included in (12·3). One might, therefore, be tempted to conclude that energy is lost under gravitational motion, but this would be unjustified for the following reason.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

To obtain the solution for free motion we must evidently ensure that there are no non-gravitational forces acting on the particles; that is, that the stresses due to springs or other machinery vanish. If in the ordinary linear approximation (corresponding to our (1*s*) approximations) one treats the particles as δ-functions and arranges that the energy tensor shall contain no stresses, then one can easily show that if the metric is flat at spatial infinity, the particles must move with constant speeds in straight lines. This is really obvious without calculation because in the linear approximation there is no interaction whatever between the particles, and any agency which accelerates them must be a non-gravitational stress. It follows that in the (1*s*) approximations what corresponds to free motion is simply unaccelerated motion, and this *is* included in (12·3). We are therefore unable to choose any function $\phi(t)$ which corresponds to free gravitational motion, and the present work gives no indication whether mass is lost by freely gravitating particles.

To deal with free gravitational motion the method of this paper would need some modification. The function $f(t)$ would have to depend on m and a, and would not be arbitrary; it would, in fact, be determined by the condition that at each approximation step the stresses must vanish. It is hard to see how this condition could be applied in practice because one would need to pick out of the $\overset{(rs)}{g}_{ik}$ terms which become singular at $r = 0$ and which correspond to the stresses along Oz. Even if this could be done for the (2*s*) approximations it would still be necessary to proceed at least to the (3*s*) approximations before terms relating to loss of mass would appear.

13. Further remarks on the method of approximation

In §7 we showed how to solve the linear approximations, that is, in our notation, the (1*s*) approximations. In §§8 to 12 we considered the non-linear, (22) approximation, and from this obtained the result of main interest, that the energy lost from the (11) waves is equal to the loss of gravitational mass which appears in the (22) approximation.

The earliest non-linear approximation is actually the (21) approximation, which may be written

$$\Phi_{lm}\left(\overset{(21)}{g}_{ik}\right) = \Psi_{lm}\left(\overset{(11)}{g}_{ik}, \overset{(10)}{g}_{ik}\right). \tag{13·1}$$

The actual equations have the form of (9·2) to (9·8) where μ, ν, σ and ρ now refer to $\overset{(21)}{g}_{ik}$, and $P, Q, \ldots, N$ are known functions of the $\overset{(11)}{g}_{ik}$ and $\overset{(10)}{g}_{ik}$. A formal solution can be achieved as described in appendix III.

It is not difficult to see that the $\overset{(21)}{g}_{ik}$ obtained by solving (13·1) will contain no terms of order r^{-1} which do not vanish at $t = \pm\infty$. The reason is that the Ψ_{lm} in (13·1) contain no terms which are quadratic in h and its derivatives. The fate of these Ψ_{lm} is in fact similar to that of the terms containing derivatives of $k(t-r)$ in the (22) approximation (see (9·25)), which produced no terms of interest. Thus the (21) approximation yields no permanent change in the mass of the system.

It will by now perhaps be clear that we should have achieved our main results if we had adopted a simpler approximation method. If we had written

$$g_{ik} = \overset{0}{g}_{ik} + \lambda \overset{1}{g}_{ik} + \lambda^2 \overset{2}{g}_{ik} + \ldots \tag{13·2}$$

with $\lambda = ma^2$, we should still have obtained the solution (10·6), which would have been simply the $\overset{2}{g}_{ik}$. It seemed worth while to use the more complicated method in order to show that combinations of terms from the (10) and (12) approximations do not contribute to the change in mass in the (22) approximation. The use of (13·2) with $\lambda = ma^2$ is certainly unsatisfactory in that it omits the Schwarzschild terms (our (10) approximation). This could be overcome by taking $\overset{0}{g}_{ik}$ to be the Schwarzschild solution, but even if one does this one still omits the (12) wave terms because these are the coefficients of ma^4 which does not appear in the expansion. There is little doubt that (13·2) would be unsatisfactory also at the non-linear approximations higher than (22).

Let us now examine the relative magnitudes of terms in the expansion (4·1). It will be convenient to think of $f(t)$ as containing a parameter ω, of dimensions t^{-1}, which serves as a 'frequency'; then for $f(t)$ one may have in mind, for example,

$$f(t) = \sin \omega t \tag{13·3}$$

remembering that this is strictly permissible only if the motion has a finite duration, and if there are smooth transitions from motion to rest, and vice versa.

Re-introducing the velocity of light c, we find for the ratio of the magnitudes of the terms in r^{-1} in the $(1s+1)$ and the $(1s)$ waves

$$m(a\omega/c)^{2s+2} \div m(a\omega/c)^{2s} = v^2/c^2,$$

where $v = a\omega$ is of the order of the maximum speed of the particles. Thus the magnitudes of the terms in r^{-1} of the first approximations will diminish rapidly if the motion is slow. We have concentrated attention mainly on the (11) wave—it is this which by carrying away energy causes the reduction of mass (10·7)—and we see now that our result will be the more accurate the slower the motion.

To illustrate the comparison of the non-linear terms with the linear ones we may take

$$m^2a^4\overset{(22)}{g}_{11} : ma^2\overset{(11)}{g}_{11} = \frac{m}{a}\left(\frac{v}{c}\right)^3,$$

where $\overset{(22)}{g}_{11}$ is given by (10·6) and m is measured in units of length. The ratio m/a is bound to be small for any real bodies.

The comparison of other terms can be made in a similar way. The result is that the series (4·1) will converge the more rapidly the smaller the ratio v/c.

Similar conclusions apply if $f(t)$ has a form different from (13·3) provided that it and its derivatives with respect to ωt do not become large. This can be arranged by taking t_1 in (12·2) not too great.

14. Conclusion

The main conclusion of this paper is contained in § 10, where it was found that the (22) approximation contains a term representing the loss of mass of a moving system equal to the energy which is carried away by the spherical quadrupole waves of the (11) approximation. There seems no reason why these particular approximations should be in any way exceptional, and it seems justifiable to suppose that an exact solution of (1·1) would give a similar result.

The work depends on certain assumptions about the motion of the particles. In the first place (§ 2) these were assumed to be moving symmetrically about their mid-point; and secondly, the function $f(t)$ describing the motion was supposed to satisfy certain mathematical conditions (§ 8). It seems certain that the relaxation of the first of these assumptions would not alter the fact that mass is lost; in regard to the second, its main disadvantage, that it requires motion to continue for an infinite time, can be overcome by using the exponential expressions of § 12. Discontinuous motions, posing boundary-value problems to which the present method is not well suited, need further study, but it would be surprising if discontinuities in $f(t)$, at the start and finish of a motion which can go on for an arbitrarily long time, were found to have any considerable effect.

A small class of motions does not involve loss of energy (up to the (22) approximation) and these are given by (12·3). As was explained in § 12, it has not been possible to find out whether energy is lost in free gravitational motion. Nor can one say from this work whether, in the motions for which energy is lost, it is the particles themselves which lose mass, or the machine which drives them.

The results of the paper depend on the choice of retarded potentials. This choice seems the most plausible, but it is not dictated by the theory of relativity: or, at least, it is not necessary if one approaches the problem with our approximation method. It is just possible that an exact solution would relate the choice of potential to boundary conditions at infinity and so remove the arbitrariness.

Perhaps the most important general conclusion to be drawn from this work is that gravitational radiation has a real existence, and like other radiation, carries energy away from the sources. As explained in the Introduction, this prediction of the linear approximation to (1·1) had become doubtful in recent years, particularly because of the deductions made from the approximation procedure of E. I. & H. If the results of this work are accepted it should be possible to study the properties and effects of gravitational radiation with the confidence that it represents a real phenomenon.

Appendix I. The retarded-potential solution

The aim here is to derive formulae (3·5), (3·9) and (3·10) starting from the expression (3·3), in which the square bracket means that the quantity inside it is to be taken at time $t-r_1/c$. The derivation follows closely a similar one of Eddington (1924, p. 253).

Consider the fixed point P at time t and the moving source A at time $t-\tau$ (figure 1). Since the motion of A is prescribed, the distance AP is given as a function of $t-\tau$

$$AP = r_1 = \phi(t-\tau).$$

(ϕ here has, of course, nothing to do with the azimuthal angle of polar co-ordinates.) The component of the velocity of A along AP is then

$$-\mathrm{d}r_1/\mathrm{d}t = -\phi'(t-\tau). \tag{I. 1}$$

Suppose that the wave emitted from A at time $t-\tau$ has at time t reached a point Q on AP, and let $PQ = u$; then

$$u = c\tau - r_1 = c\tau - \phi(t-\tau), \tag{I. 2}$$

and u is a function of τ and vice versa. Differentiating (I. 2) we have

$$1 = c\frac{d\tau}{du} + \phi'(t-\tau)\frac{d\tau}{du}.$$

Using this, and (I. 1), we have

$$\frac{m}{\{r_1-(\mathbf{v}_1\cdot\mathbf{r}_1)/c\}_{t-\tau}} = \frac{mc}{\phi(t-\tau)}\frac{d\tau}{du} = cm\frac{d}{du}\Phi(t-\tau),$$

where $$\Phi' = -1/\phi.$$

To calculate the retarded potential we need Q to coincide with P; this is achieved by putting $u=0$; then

$$\frac{m}{[r_1-(\mathbf{v}_1\cdot\mathbf{r}_1)/c]} = mc\left\{\frac{d}{du}\Phi(t-\tau)\right\}_{u=0}. \tag{I. 3}$$

We now use Lagrange's theorem on implicit functions (Whittaker & Watson 1952, p. 133). Write (I. 2) in the form

$$t-\tau = \left(t-\frac{u}{c}-\frac{r}{c}\right)-\frac{1}{c}g(t-\tau),$$

where the function g has been introduced from (3·4) and r is the fixed radius vector OP. Then the theorem gives

$$\Phi(t-\tau) = \Phi(w)+\sum_{n=1}^{\infty}\left(-\frac{1}{c}\right)^n\frac{1}{n!}\frac{d^{n-1}}{dw^{n-1}}\{\Phi'(w)\,[g(w)]^n\}, \tag{I. 4}$$

where $$w = t-\frac{u}{c}-\frac{r}{c}.$$

Substituting (I. 4) into (I. 3), and using the fact that $d/dw = -c\,\partial/\partial u = \partial/\partial t$, we obtain (3·5)

$$\frac{m}{[r_1-(\mathbf{v}_1\cdot\mathbf{r}_1)/c]} = \frac{m}{r_1}+m\sum_{n=1}^{\infty}\left(-\frac{1}{c}\right)^n\frac{1}{n!}\frac{\partial^n}{\partial t^n}\left(\frac{g^n}{r_1}\right), \tag{3·5}$$

where r_1 and g are to be calculated at time $t-r/c$.

We now use the binomial theorem to expand r_1 and g given by (2·1) and (3·4), in powers of r^{-1}. For example, we obtain

$$\frac{g}{r_1} = -\left(\frac{af}{r}P_1+\frac{a^2f^2}{r^2}P_2+\frac{a^3f^3}{r^3}P_3+\dots\right),$$

$$\frac{g^2}{r_1} = \frac{a^2f^2}{r}\left\{(\tfrac{2}{3}P_2+\tfrac{1}{3})+\frac{af}{r}(\tfrac{4}{5}P_3+\tfrac{1}{5}P_1)+\frac{a^2f^2}{r^2}(\tfrac{6}{7}P_4+\tfrac{1}{7}P_2)+\dots\right\},$$

in which f is to be taken at time $t-r/c$. Using expressions such as these in (3·5), we can collect the various terms together in the form of a series (3·7). This series is then added to the corresponding one for the second particle, obtained in a similar way, and the result is (3·8), with G_2 and G_4 given by (3·9) and (3·10).

Appendix II. The field equations

The non-zero Christoffel symbols for the metric (5·8), to the approximation required, are given below. The notation of (6·1) is used, and the sign Σ denotes summation for s over the values 1 and 2. The expressions (7·1) and (7·2) for the $\overset{(00)}{g}_{ik}$ and $\overset{(10)}{g}_{ik}$ have already been inserted, so that $\overset{(10)}{A}$, $\overset{(10)}{B}$, $\overset{(10)}{C}$ and $\overset{(10)}{D}$ do not appear below.

$$\Gamma^1_{11} = -2mr^{-2} + \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{A}_1 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{A}_1 - \tfrac{1}{2}\overset{(11)}{A}\,\overset{(11)}{A}_1 - 2r^{-1}\overset{(12)}{A}_1 + 2r^{-2}\overset{(12)}{A}\right),$$

$$\Gamma^1_{12} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{A}_2 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{A}_2 - \tfrac{1}{2}\overset{(11)}{A}\,\overset{(11)}{A}_2 - 2r^{-1}\overset{(12)}{A}_2\right),$$

$$\Gamma^1_{14} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{A}_4 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{A}_4 - \tfrac{1}{2}\overset{(11)}{A}\,\overset{(11)}{A}_4 - 2r^{-1}\overset{(12)}{A}_{4,}\right),$$

$$\Gamma^1_{22} = -r + 4m + \Sigma ma^{2s}\left(r\overset{(1s)}{A} - r\overset{(1s)}{B} - \tfrac{1}{2}r^2\overset{(1s)}{B}_1\right)$$
$$+ m^2a^4\left(r\overset{(22)}{A} - r\overset{(22)}{B} - \tfrac{1}{2}r^2\overset{(22)}{B}_1 + r\overset{(11)}{A}\,\overset{(11)}{B} + \tfrac{1}{2}r^2\overset{(11)}{A}\,\overset{(11)}{B}_1 - r\overset{(11)}{A}{}^2 - 8\overset{(12)}{A} + 4\overset{(12)}{B} + 2r\overset{(12)}{B}_{1,}\right),$$

$$\Gamma^1_{33} = \sin^2\theta\Big\{-r + 4m + \Sigma ma^{2s}\left(r\overset{(1s)}{A} - r\overset{(1s)}{C} - \tfrac{1}{2}r^2\overset{(1s)}{C}_1\right)$$
$$+ m^2a^4\left(r\overset{(22)}{A} - r\overset{(22)}{C} - \tfrac{1}{2}r^2\overset{(22)}{C}_1 + r\overset{(11)}{A}\,\overset{(11)}{C} + \tfrac{1}{2}r^2\overset{(11)}{A}\,\overset{(11)}{C}_1 - r\overset{(11)}{A}{}^2 - 8\overset{(12)}{A} + 4\overset{(12)}{C} + 2r\overset{(12)}{C}_1\right)\Big\},$$

$$\Gamma^1_{44} = 2mr^{-2} + \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{D}_1 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{D}_1 - \tfrac{1}{2}\overset{(11)}{A}\,\overset{(11)}{D}_1 - 2r^{-1}\overset{(12)}{D}_1 - 2r^{-2}\overset{(12)}{A}\right);$$

$$\Gamma^2_{11} = -\tfrac{1}{2}\Sigma ma^{2s}r^{-2}\overset{(1s)}{A}_2 + \tfrac{1}{2}m^2a^4\left(r^{-2}\overset{(11)}{A}_2\,\overset{(11)}{B} - r^{-2}\overset{(22)}{A}_2\right),$$

$$\Gamma^2_{12} = r^{-1} + \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{B}_1 + \tfrac{1}{2}m^2a^4\left(\overset{(22)}{B}_1 - \overset{(11)}{B}\,\overset{(11)}{B}_1\right),$$

$$\Gamma^2_{22} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{B}_2 + \tfrac{1}{2}m^2a^4\left(\overset{(22)}{B}_2 - \overset{(11)}{B}\,\overset{(11)}{B}_2\right),$$

$$\Gamma^2_{24} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{B}_4 + \tfrac{1}{2}m^2a^4\left(\overset{(22)}{B}_4 - \overset{(11)}{B}\,\overset{(11)}{B}_4\right),$$

$$\Gamma^2_{33} = \sin^2\theta\Big\{-\cot\theta + \Sigma ma^{2s}\left[\left(\overset{(1s)}{B} - \overset{(1s)}{C}\right)\cot\theta - \tfrac{1}{2}\overset{(1s)}{C}_2\right]$$
$$+ m^2a^4\left[\left(\overset{(22)}{B} - \overset{(22)}{C}\right)\cot\theta - \tfrac{1}{2}\overset{(22)}{C}_2 + \left(\overset{(11)}{C}\,\overset{(11)}{B} - \overset{(11)}{B}{}^2\right)\cot\theta + \tfrac{1}{2}\overset{(11)}{B}\,\overset{(11)}{C}_2\right]\Big\},$$

$$\Gamma^2_{44} = \tfrac{1}{2}\Sigma ma^{2s}r^{-2}\overset{(1s)}{D}_2 + \tfrac{1}{2}m^2a^4r^{-2}\left(\overset{(22)}{D}_2 - \overset{(11)}{B}\,\overset{(11)}{D}_2\right);$$

$$\Gamma^3_{13} = r^{-1} + \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{C}_1 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{C}_1 - \tfrac{1}{2}\overset{(11)}{C}\,\overset{(11)}{C}_1\right),$$

$$\Gamma^3_{23} = \cot\theta + \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{C}_2 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{C}_2 - \tfrac{1}{2}\overset{(11)}{C}\,\overset{(11)}{C}_2\right),$$

$$\Gamma^3_{34} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{C}_4 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{C}_4 - \tfrac{1}{2}\overset{(11)}{C}\,\overset{(11)}{C}_4\right);$$

$$\Gamma^4_{11} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{A}_4 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{A}_4 - \tfrac{1}{2}\overset{(11)}{D}\,\overset{(11)}{A}_4 + 2r^{-1}\overset{(12)}{A}_4\right),$$

$$\Gamma^4_{14} = 2mr^{-2} + \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{D}_1 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{D}_1 - \tfrac{1}{2}\overset{(11)}{D}\,\overset{(11)}{D}_1 + 2r^{-1}\overset{(12)}{D}_1 - 2r^{-2}\overset{(12)}{D}\right),$$

$$\Gamma^4_{22} = \tfrac{1}{2}\Sigma ma^{2s}r^2\overset{(1s)}{B}_4 + m^2a^4\left(\tfrac{1}{2}r^2\overset{(22)}{B}_4 - \tfrac{1}{2}r^2\overset{(11)}{D}\,\overset{(11)}{B}_4 + 2r\overset{(12)}{B}_4\right),$$

$$\Gamma^4_{24} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{D}_2 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{D}_2 - \tfrac{1}{2}\overset{(11)}{D}\,\overset{(11)}{D}_2 + 2r^{-1}\overset{(12)}{D}_2\right),$$

$$\Gamma^4_{33} = r^2\sin^2\theta\Big\{\tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{C}_4 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{C}_4 - \tfrac{1}{2}\overset{(11)}{D}\,\overset{(11)}{C}_4 + 2r^{-1}\overset{(12)}{C}_4\right)\Big\},$$

$$\Gamma^4_{44} = \tfrac{1}{2}\Sigma ma^{2s}\overset{(1s)}{D}_4 + m^2a^4\left(\tfrac{1}{2}\overset{(22)}{D}_4 - \tfrac{1}{2}\overset{(11)}{D}\,\overset{(11)}{D}_4 + 2r^{-1}\overset{(12)}{D}_4\right).$$

In the above, contributions from $\overset{(00)}{g}_{ik}$, $\overset{(10)}{g}_{ik}$, $\overset{(11)}{g}_{ik}$, $\overset{(12)}{g}_{ik}$, $\overset{(22)}{g}_{ik}$ have been included so that Γ^i_{jk} is complete in the following terms

$$\Gamma^i_{jk} = \overset{(00)}{\Gamma}{}^i_{jk} + m\overset{(10)}{\Gamma}{}^i_{jk} + ma^2\overset{(11)}{\Gamma}{}^i_{jk} + ma^4\overset{(12)}{\Gamma}{}^i_{jk} + m^2a^4\overset{(22)}{\Gamma}{}^i_{jk}. \tag{II. 1}$$

To find the field equations we have to substitute Γ^i_{jk} into

$$R_{ik} \equiv \Gamma^a_{ia,k} - \Gamma^a_{ik,a} + \Gamma^a_{ib}\Gamma^b_{ka} - \Gamma^a_{ik}\Gamma^b_{ab} = 0.$$

From (II. 1) it follows that to our degree of approximation R_{ik} will contain the following terms

$$R_{ik} = \overset{(00)}{R}_{ik} + m\overset{(10)}{R}_{ik} + ma^2\overset{(11)}{R}_{ik} + ma^4\overset{(12)}{R}_{ik} + m^2a^4\overset{(22)}{R}_{ik}. \tag{II. 2}$$

Each of the terms on the right of (II. 2) is to be put equal to zero. When this is done, $\overset{(00)}{R}_{ik} = 0$ and $\overset{(10)}{R}_{ik} = 0$ are satisfied identically, and both sets of $\overset{(1s)}{R}_{ik} = 0$ $(s = 1, 2)$ give equations (7·3) to (7·9) in which, as explained in the text, α, β, γ, and δ are used to denote $\overset{(1s)}{A}$, $\overset{(1s)}{B}$, $\overset{(1s)}{C}$ and $\overset{(1s)}{D}$.

The equations of the non-linear (22) approximation come from

$$\overset{(22)}{R}_{ik} = 0, \tag{II. 3}$$

and are (9·2) to (9·8), using the notation of (6·3). The right-hand sides of these equations are given, up to terms in r^{-2}, by (9·25), and to see how these have been obtained we shall consider one particular member of (II. 3), $\overset{(22)}{R}_{11} = 0$, in detail as an example. Using the Christoffel symbols, one finds

$$\begin{aligned} 2\overset{(22)}{R}_{11} = {} & \Big[\overset{(22)}{B}_{11} + \overset{(22)}{C}_{11} + \overset{(22)}{D}_{11} + 2r^{-1}\Big(\overset{(22)}{B}_1 + \overset{(22)}{C}_1 - \overset{(22)}{A}_1\Big) + r^{-2}\Big(\overset{(22)}{A}_{22} + \overset{(22)}{A}_2\cot\theta\Big) - \overset{(22)}{A}_{44}\Big] \\ & + 2\Big[r^{-2}\Big(3\overset{(12)}{A}_1 + \overset{(12)}{B}_1 + \overset{(12)}{C}_1 - \overset{(12)}{D}_1\Big) + 2r^{-1}\Big(\overset{(12)}{D}_{11} - \overset{(12)}{A}_{44}\Big) + 4r^{-3}\Big(\overset{(12)}{D} - \overset{(12)}{A}\Big)\Big] \\ & + \Big[-\Big(\overset{(11)}{B}\overset{(11)}{B}_{11} + \overset{(11)}{C}\overset{(11)}{C}_{11} + \overset{(11)}{D}\overset{(11)}{D}_{11}\Big) - \tfrac{1}{2}\Big(\overset{(11)}{B}_1\overset{(11)}{B}_1 + \overset{(11)}{C}_1\overset{(11)}{C}_1 + \overset{(11)}{D}_1\overset{(11)}{D}_1\Big) \\ & - \tfrac{1}{2}\Big(\overset{(11)}{A}_1\overset{(11)}{B}_1 + \overset{(11)}{A}_1\overset{(11)}{C}_1 + \overset{(11)}{A}_1\overset{(11)}{D}_1\Big) + 2r^{-1}\Big(\overset{(11)}{A}\overset{(11)}{A}_1 - \overset{(11)}{B}\overset{(11)}{B}_1 - \overset{(11)}{C}\overset{(11)}{C}_1\Big) \\ & + \tfrac{1}{2}\overset{(11)}{A}_4\Big(\overset{(11)}{A}_4 - \overset{(11)}{B}_4 - \overset{(11)}{C}_4 + \overset{(11)}{D}_4\Big) + \overset{(11)}{A}_{44}\overset{(11)}{D} \\ & + r^{-2}\Big(\tfrac{1}{2}\overset{(11)}{A}_2\overset{(11)}{C}_2 + \tfrac{1}{2}\overset{(11)}{A}_2\overset{(11)}{D}_2 - \tfrac{1}{2}\overset{(11)}{A}_2\overset{(11)}{A}_2 - \tfrac{1}{2}\overset{(11)}{A}_2\overset{(11)}{B}_2 - \overset{(11)}{A}_{22}\overset{(11)}{B}\Big) - r^{-2}\overset{(11)}{A}_2\overset{(11)}{B}\cot\theta\Big]. \end{aligned} \tag{II. 4}$$

The terms in the first square bracket on the right of (II. 4) give the left-hand side of (9·2), and the remainder constitute P on the right-hand side. Of these we need, as explained in § 9, only those which are of order r^{-2}. The appropriate values of $\overset{(11)}{A}, \ldots, \overset{(12)}{A}, \ldots,$ to be inserted in the second and third square brackets of (II. 4) are therefore those given by (7·12) and (7·14) because it is clear that terms of order r^{-2} and higher in $\overset{(11)}{A}, \ldots, \overset{(12)}{A}, \ldots,$ will not contribute to the term in r^{-2} in P. Inserting the solutions (7·12) and (7·14) we find that the only term of order r^{-2} in P is

$$r^{-2}X\sin^4\theta,$$

in agreement with (9·25). The remaining equations (9·3) to (9·8) are obtained similarly.

Appendix III. Solution of the approximate field equations

The object here is to find solutions of the (1*s*) and (22) approximations. The equations of the (22) approximation are (9·2) to (9·8), and indeed the equations of *any* approximation step whatever can be written as (9·2) to (9·8), with appropriate values for $P, Q, \ldots, N$, which are always known from the previous approximations. We shall therefore give a formal solution of (9·2) to (9·8); the solution can then be used for the (1*s*) approximations by putting $P, Q, \ldots, N$ all zero.

Integrating (9·7) and (9·8) we have

$$\nu_1+\sigma_1 = r^{-1}(2\mu-\nu-\sigma)+u(r,\theta)+\int M\,\mathrm{d}t, \tag{III. 1}$$

$$\mu_2+\sigma_2 = \cot\theta(\nu-\sigma)+v(r,\theta)+\int N\,\mathrm{d}t, \tag{III. 2}$$

where u and v are functions of integration. Substituting (III. 2) into (9·6) and integrating we find

$$\rho_1-\mu_1 = r^{-1}(\rho+\mu)+\int L\,\mathrm{d}\theta-\int\left\{\int N_1\,\mathrm{d}t\right\}\mathrm{d}\theta-\int v_1\,\mathrm{d}\theta+w(r,t), \tag{III. 3}$$

where w is a function of integration.

If we now substitute (III. 1) and (III. 3) into (9·2) we get

$$\begin{aligned}\mu_{11}+2r^{-1}\mu_1+r^{-2}(\mu_{22}+\mu_2\cot\theta)-\mu_{44}&\\ = P-\int(L_1+r^{-1}L)\,\mathrm{d}\theta-\int(M_1+r^{-1}M)\,\mathrm{d}t+\int\left\{\int(N_{11}+r^{-1}N_1)\,\mathrm{d}t\right\}\mathrm{d}\theta&\\ +\int(v_{11}+r^{-1}v_1)\,\mathrm{d}\theta-(u_1+r^{-1}u)-(w_1+r^{-1}w).&\end{aligned} \tag{9·9}$$

This is the inhomogeneous wave equation, and a formal solution of it can be given in terms of retarded potentials in the usual way.

Equations (III. 1), (III. 2) and (III. 3) enable us to write down expressions for ν, σ and ρ in terms of μ. From (III. 3) we find

$$\rho = \mu+2r\int r^{-2}\mu\,\mathrm{d}r+r\int\left\{\int r^{-1}L\,\mathrm{d}\theta-\int\left[\int r^{-1}N_1\,\mathrm{d}t\right]\mathrm{d}\theta-\int r^{-1}v_1\,\mathrm{d}\theta+r^{-1}w\right\}\mathrm{d}r+rb(\theta,t), \tag{III. 4}$$

where b is another function of integration.

Equation (III. 1) gives on integration with respect to r

$$\nu+\sigma = 2r^{-1}\int\mu\,\mathrm{d}r+r^{-1}\int ru\,\mathrm{d}r+r^{-1}\int\left[\int rM\,\mathrm{d}t\right]\mathrm{d}r+r^{-1}d(\theta,t), \tag{III. 5}$$

where d is yet another function of integration. Finally, we eliminate ν between (III. 2) and (III. 5); this gives

$$\sigma_2 = -\mu_2+\cot\theta\left\{-2\sigma+2r^{-1}\int\mu\,\mathrm{d}r+r^{-1}\int\left[\int rM\,\mathrm{d}t\right]\mathrm{d}r+r^{-1}\int ru\,\mathrm{d}r+r^{-1}d\right\}+v+\int N\,\mathrm{d}t,$$

whence we find

$$\begin{aligned}\sigma = -\mu+\operatorname{cosec}^2\theta\int\left\{\sin\theta\cos\theta\left[2\mu+2r^{-1}\int\mu\,\mathrm{d}r+r^{-1}\int\left(\int rM\,\mathrm{d}t\right)\mathrm{d}r+r^{-1}\int ru\,\mathrm{d}r+r^{-1}d\right]\right\}\mathrm{d}\theta&\\ +\operatorname{cosec}^2\theta\int v\sin^2\theta\,\mathrm{d}\theta+q(r,t)\operatorname{cosec}^2\theta+\operatorname{cosec}^2\theta\int\left[\sin^2\theta\int N\,\mathrm{d}t\right]\mathrm{d}\theta,&\end{aligned} \tag{III. 6}$$

q being a final function of integration.

The function ν may now be obtained from (III. 5) and (III. 6) and is

$$\nu = \mu + 2r^{-1}\int \mu\,\mathrm{d}r + r^{-1}\int ru\,\mathrm{d}r + r^{-1}\int\left(\int rM\,\mathrm{d}t\right)\mathrm{d}r + r^{-1}d$$
$$-\operatorname{cosec}^2\theta\int\left\{\sin\theta\cos\theta\left[2\mu+2r^{-1}\int\mu\,\mathrm{d}r + r^{-1}\int\left(\int rM\,\mathrm{d}t\right)\mathrm{d}r + r^{-1}\int ru\,\mathrm{d}r + r^{-1}d\right]\right\}\mathrm{d}\theta$$
$$-\operatorname{cosec}^2\theta\int\sin^2\theta\left[v+\int N\,\mathrm{d}t\right]\mathrm{d}\theta - q\operatorname{cosec}^2\theta. \qquad \text{(III. 7)}$$

The complete solution of any given approximation step is made up of μ, obtained from (9·9), and the values of ν, σ and ρ, obtained from (III. 7), (III. 6) and (III. 4). The solution contains six functions of integration, $u(r,\theta)$, $v(r,\theta)$, $w(r,t)$, $b(\theta,t)$, $d(\theta,t)$ and $q(r,t)$. The field equations may impose certain relations between these functions, and it is necessary to verify that a solution using them actually satisfies (9·2) to (9·8).

For the (1*s*) approximations (§ 7) the foregoing work applies if we put

$$\mu = \alpha, \quad \nu = \beta, \quad \sigma = \gamma, \quad \rho = \delta,$$
$$P = Q = R = S = L = M = N = 0.$$

The equations determining the solution then become

$$\alpha_{11} + 2r^{-1}\alpha_1 + r^{-2}(\alpha_{22}+\alpha_2\cot\theta) - \alpha_{44} = \int(v_{11}+r^{-1}v_1)\,\mathrm{d}\theta - (u_1+r^{-1}u) - (w_1+r^{-1}w). \qquad (7\cdot 10)$$

$$\delta = \alpha + 2r\int r^{-2}\alpha\,\mathrm{d}r - r\int\left[\int r^{-1}v_1\,\mathrm{d}\theta - r^{-1}w\right]\mathrm{d}r + rb, \qquad \text{(III. 8)}$$

$$\beta = \alpha + 2r^{-1}\int\alpha\,\mathrm{d}r + r^{-1}\int ru\,\mathrm{d}r + r^{-1}d - \operatorname{cosec}^2\theta\int v\sin^2\theta\,\mathrm{d}\theta - q\operatorname{cosec}^2\theta$$
$$-\operatorname{cosec}^2\theta\int\left\{\sin\theta\cos\theta\left[2\alpha+2r^{-1}\int\alpha\,\mathrm{d}r + r^{-1}\int ru\,\mathrm{d}r + r^{-1}d\right]\right\}\mathrm{d}\theta, \qquad \text{(III. 9)}$$

$$\gamma = -\alpha + \operatorname{cosec}^2\theta\int v\sin^2\theta\,\mathrm{d}\theta + q\operatorname{cosec}^2\theta$$
$$+\operatorname{cosec}^2\theta\int\left\{\sin\theta\cos\theta\left[2\alpha+2r^{-1}\int\alpha\,\mathrm{d}r + r^{-1}\int ru\,\mathrm{d}r + r^{-1}d\right]\right\}\mathrm{d}\theta. \qquad \text{(III. 10)}$$

To find the solution of the (11) approximation we first take α to be equal to $-8G_2$ (see below), where G_2 is given by (3·9) (with $c = 1$). This satisfies (7·10) with $u = v = w = 0$. Then we substitute this value of α into (III. 8), (III. 9) and (III. 10), carry out partial integrations and simplify; this leads to the expressions for $\overset{(11)}{B}$, $\overset{(11)}{C}$ and $\overset{(11)}{D}$ in (7·11), the other functions of integration being chosen so that the integrals remaining after partial integration have the limits shown. One may check by direct substitution that the set (7·11) satisfies the field equations (7·3) to (7·9). In the course of this verification it is necessary to differentiate the integrals in (7·11) with respect to t through the sign of integration. This is in order because, as is apparent from (9·11), the integrals are uniformly convergent for all finite values of t, if $r > 0$.

In the full (12) approximation we should take $\alpha = -6G_4$ (see below), G_4 being given by (3·10) (with $c = 1$). However, we need only terms in r^{-1} in the solution of the (12) approximation, and it is sufficient to take

$$\begin{aligned}\alpha &= -6r^{-1}(f^4)^{\text{iv}}[\tfrac{1}{120}+\tfrac{1}{42}P_2+\tfrac{1}{105}P_4] \\ &= -\tfrac{1}{4}r^{-1}(f^4)^{\text{iv}} \quad \cos^4\theta.\end{aligned}$$

Substituting this into (III.8), (III.9) and (III.10), and taking $q = \frac{1}{18}r^{-1}(f^4)^{\text{iv}}$, we find the solution (7·14). By direct substitution this set of expressions satisfies the field equations (7·3) to (7·9) to order r^{-1}.

An explanation will now be given of the factor -8 in $\alpha = -8G_2$. The procedure will be to show that the equations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik}, \tag{III.11}$$

where T_{ik} is the energy tensor, have an approximate solution which, when the sources are two symmetrically oscillating particles of mass m and separation $2af(t)$, is the same as (7·12) provided that one takes (7·13) for h. This amounts to the above choice for α.

Let us write

$$g_{ik} = e_i\delta_{ik} + \gamma_{ik},$$

where e_i takes the value -1 when $i = 1, 2, 3$ and $+1$ when $i = 4$, and where the γ_{ik} are small. Introduce, as usual in the weak field theory,

$$\gamma^*_{ik} = \gamma_{ik} - \tfrac{1}{2}e_i\delta_{ik}e_j\gamma_{jj}, \tag{III.12}$$

where

$$e_j\gamma_{jj} = -\gamma_{11} - \gamma_{22} - \gamma_{33} + \gamma_{44},$$

and choose co-ordinates so that

$$e_k\gamma^*_{ik,\,k} = 0. \tag{III.13}$$

Then (III.11) reduces to

$$e_a\gamma^*_{ik,\,aa} = -16\pi T_{ik}. \tag{III.14}$$

Taking the divergence of (III.14) and using (III.13) we find that T_{ik} must satisfy

$$e_k T_{ik,\,k} = 0.$$

We now use a standard procedure on this equation (Landau & Lifshitz 1951, p. 329), and derive

$$\int T_{\alpha\beta}\,\mathrm{d}V = \frac{1}{2}\frac{\partial^2}{\partial x_4^2}\int T_{44}x_\alpha x_\beta\,\mathrm{d}V \quad (\alpha, \beta = 1, 2, 3), \tag{III.15}$$

where the integrals are to be taken over all space.

Integrating (III.14) in terms of retarded potentials and supposing that the sources are located within a region small compared with the distance r of the field-point, we find

$$\gamma^*_{\alpha\beta} = -\frac{4}{r}\int [T_{\alpha\beta}]\,\mathrm{d}V, \tag{III.16}$$

where the square brackets mean that $T_{\alpha\beta}$ is to be calculated at the time $t-r$ ($c = 1$). From (III.15) and (III.16) we have

$$\gamma^*_{\alpha\beta} = -\frac{2}{r}\frac{\partial^2}{\partial t^2}\int \rho x_\alpha x_\beta\,\mathrm{d}V, \tag{III.17}$$

where ρ is the mass density, calculated at the retarded time. Landau & Lifshitz (1951, p. 330) give this formula, but with the wrong sign.

In our problem the particles are located on the z-axis and the integral in (III. 17) is I, the moment of inertia about $z = 0$, when $\alpha = \beta = 3$; for any other pair of values of α and β it vanishes. Therefore

$$\gamma^*_{33} = -\frac{2}{r}\frac{\partial^2}{\partial t^2} I(t-r).$$

The remaining γ^*_{ik} are found from (III. 13) and are (up to order r^{-1})

$$\gamma^*_{44} = -\frac{2z^2}{r^3}\ddot{I}, \quad \gamma^*_{34} = \frac{2z}{r^2}\ddot{I},$$

where $\cdot$ means $\partial/\partial t$.

We now reinstate the γ_{ik} by (III. 12), and then transform the resulting metric to polar co-ordinates. The solution in polar co-ordinates contains non-zero coefficients g_{12}, g_{14}, g_{24}, but by means of a transformation of type (5·2) it can be brought into the form (7·12), provided that

$$ma^2h = -2\ddot{I},$$

which is equivalent to (7·13), and which requires

$$\alpha \equiv \overset{(11)}{A} = -8G_2.$$

As a further check on the factor -8 we may consider the case when f is constant, so that we find from (7·11)

$$ma^2\overset{(11)}{D} = -4ma^2f^2r^{-3}P_2.$$

This is the correct form for $\overset{(11)}{D}$ because in the static case g_{44} represents the Newtonian potential

$$g_{44} = 1 - 2V,$$

and the term in V which involves the moment of inertia of the particles is

$$r^{-3}IP_2 = 2ma^2f^2r^{-3}P_2.$$

For the (12) approximation we take $\alpha = -6G_4$. In this case it is not so easy to derive the factor from approximate solutions of (III. 11), and it was obtained instead by the second method of comparing the corresponding static solution with the Newtonian potential. This factor is in any case less important as the (12) solution does not contribute to the loss of mass of the system derived in §§ 9 and 10.

References

Bach, R. & Weyl, H. 1922 *Math. Z.* **13**, 142.
Bondi, H. 1957 *Nature, Lond.* **179**, 1072.
Bonnor, W. B. 1957 *J. Math. Mech.* **6**, 203.
Eddington, A. S. 1924 *The mathematical theory of relativity.* Cambridge University Press.
Einstein, A., Infeld, L. & Hoffmann, B. 1938 *Ann. Math.* **39**, 65.
Einstein, A. & Rosen, N. 1937 *J. Franklin Inst.* **223**, 43.
Fock, V. 1957 *Rev. Mod. Phys.* **29**, 325.
Landau, L. & Lifshitz, E. 1951 *The classical theory of fields.* Cambridge, Mass: Addison-Wesley Press, Inc.

Lichnerowicz, A. 1955 *Théories Relativistes de la Gravitation et de L'Électromagnétisme*. Paris: Masson et Cie.

O'Brien, S. & Synge, J. L. 1952 *Commun. Dublin Inst. Adv. Stud.* A, no. 9.

Rosen, N. & Shamir, H. 1957 *Rev. Mod. Phys.* **29**, 429.

Takeno, H. 1956 *Tensor* (New Series), **6**, 15.

Weber, J. & Wheeler, J. A. 1957 *Rev. Mod. Phys.* **29**, 509.

Weyl, H. 1917 *Ann. Phys., Lpz.* **54**, 117.

Whittaker, E. T. & Watson G. N. 1952 *Modern analysis.* Cambridge University Press.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES